Robust chaos in piecewise-linear maps

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Importance

1. Piecewise-linear maps have applications in designing secure encryption schemes (one of the main motivations that drives me!).

2. Investigating chaos regions might let engineers design proper fail-safes in switched systems in avionics, for example!

3. Prevention of undesirable chaotic regimes while designing DC-DC power converters and inverters.

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4. Many more "…, which this margin is too small to contain."

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Border-collision normal form (BCNF)

1. In our project we study the 2D BCNF (Nusse & Yorke, 1992)

$$f_\xi(x,y) = \begin{cases} \begin{bmatrix} \tau_L & 1 \\ -\delta_L & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}, & x \le 0, \\ \begin{bmatrix} \tau_R & 1 \\ -\delta_R & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}, & x \ge 0. \end{cases}$$

2. Here $(x,y) \in \mathbb{R}^2$ and $\xi = (\tau_L, \delta_L, \tau_R, \delta_R) \in \mathbb{R}^4$ are the parameters.

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2. Here $(x,y) \in \mathbb{R}^2$ and $\xi = (\tau_L, \delta_L, \tau_R, \delta_R) \in \mathbb{R}^4$ are the parameters.

3. Any continuous, two-piece, piecewise-linear map on $\mathbb{R}^2$ satisfying a certain non-degeneracy condition can be converted to it.

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Phase portrait of a chaotic attractor

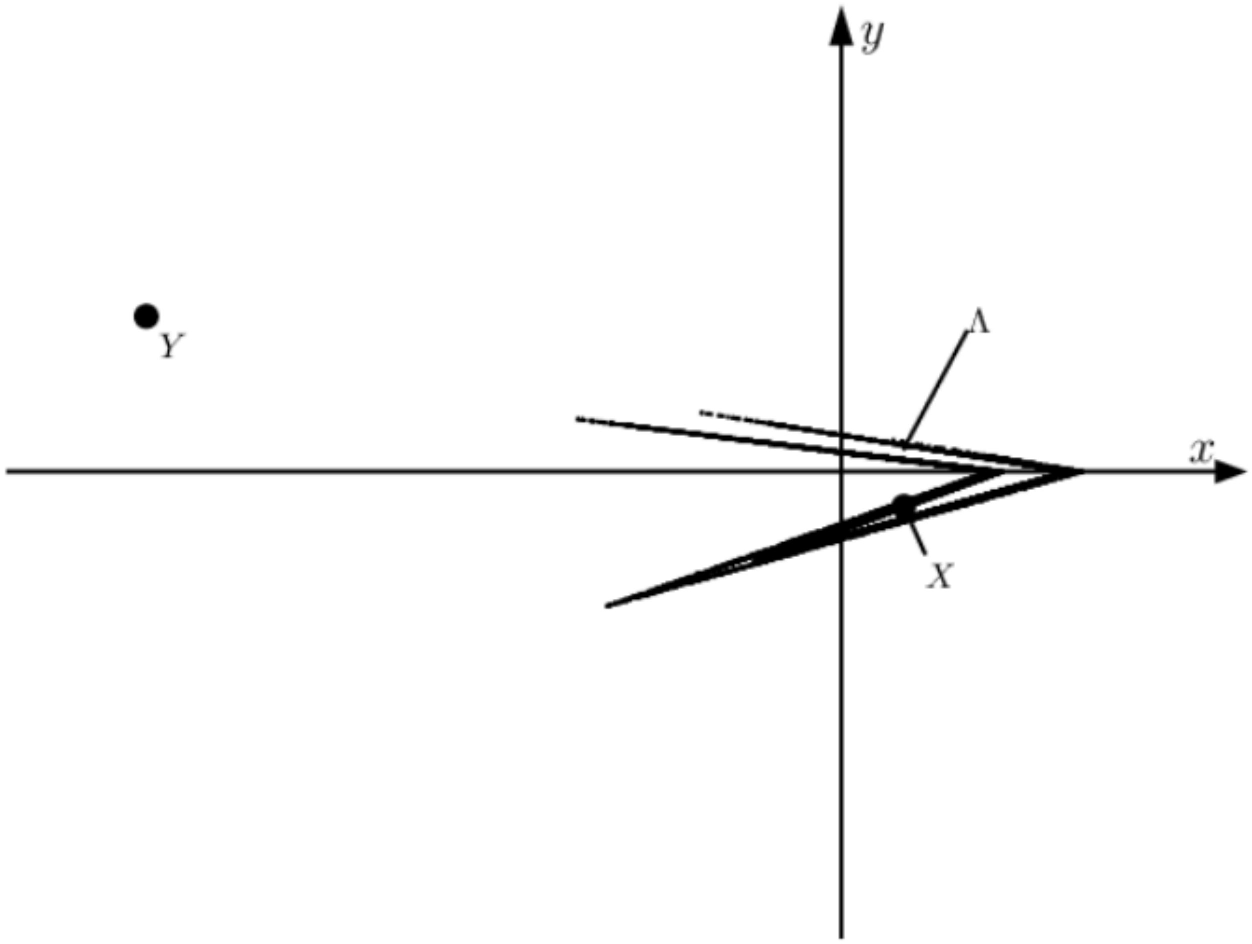

Figure: A sketch of the phase portrait of f_ξ with $\xi \in \Phi_{\mathrm{BYG}}$

Renormalisation operator

1. Renormalisation: for some members of a family of maps, a higher iterate or induced map is conjugate to different member of this family.

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1. Renormalisation: for some members of a family of maps, a higher iterate or induced map is conjugate to different member of this family.

2. Although the second iterate f_ξ^2 has four pieces, relevant dynamics only occurs in two of these:

$$f_\xi^2(x,y) = \begin{cases} \begin{bmatrix} \tau_L\tau_R - \delta_L & \tau_R \\ -\delta_R\tau_L & -\delta_R \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} \tau_R + 1 \\ -\delta_R \end{bmatrix}, & x \le 0, \\ \begin{bmatrix} \tau_R^2 - \delta_R & \tau_R \\ -\delta_R\tau_R & -\delta_R \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} \tau_R + 1 \\ -\delta_R \end{bmatrix}, & x \ge 0. \end{cases}$$

Renormalisation operator

3. Now f_{ξ}^{2} can be transformed to $f_{g(\xi)}$, where g is the renormalisation operator (Ghosh and Simpson, 2022) $g : \mathbb{R}^4 \mapsto \mathbb{R}^4$, given by

$$(\tilde{\tau}_L, \tilde{\delta}_L, \tilde{\tau}_R, \tilde{\delta}_R) = (\tau_R^2 - 2\delta_R,\ \delta_R^2,\ \tau_L\tau_R - \delta_L - \delta_R,\ \delta_L\delta_R)\,.$$

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4. We perform a coordinate change to put f_ξ^2 in the normal form:

$$\begin{bmatrix} \tilde{x}' \\ \tilde{y}' \end{bmatrix} = \begin{cases} \begin{bmatrix} \tilde{\tau}_L & 1 \\ -\tilde{\delta}_L & 0 \end{bmatrix} \begin{bmatrix} \tilde{x} \\ \tilde{y} \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}, & \tilde{x} \le 0, \\ \begin{bmatrix} \tilde{\tau}_R & 1 \\ -\tilde{\delta}_R & 0 \end{bmatrix} \begin{bmatrix} \tilde{x} \\ \tilde{y} \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}, & \tilde{x} \ge 0. \end{cases}$$

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Results

1. We consider the parameter region

$$\Phi = \{\, \xi \in \mathbb{R}^4 \mid \tau_L > \delta_L + 1,\ \delta_L > 0,\ \tau_R < -(\delta_R + 1),\ \delta_R > 0 \,\}\,.$$

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4. The attractor is often destroyed at $\phi^+(\xi) = 0$: homoclinic bifurcation (Banerjee, Yorke & Grebogi, 1998). Thus focused on the region

$$\Phi_{\mathrm{BYG}} = \{\, \xi \in \Phi \mid \phi^+(\xi) > 0 \,\}\,.$$

Results

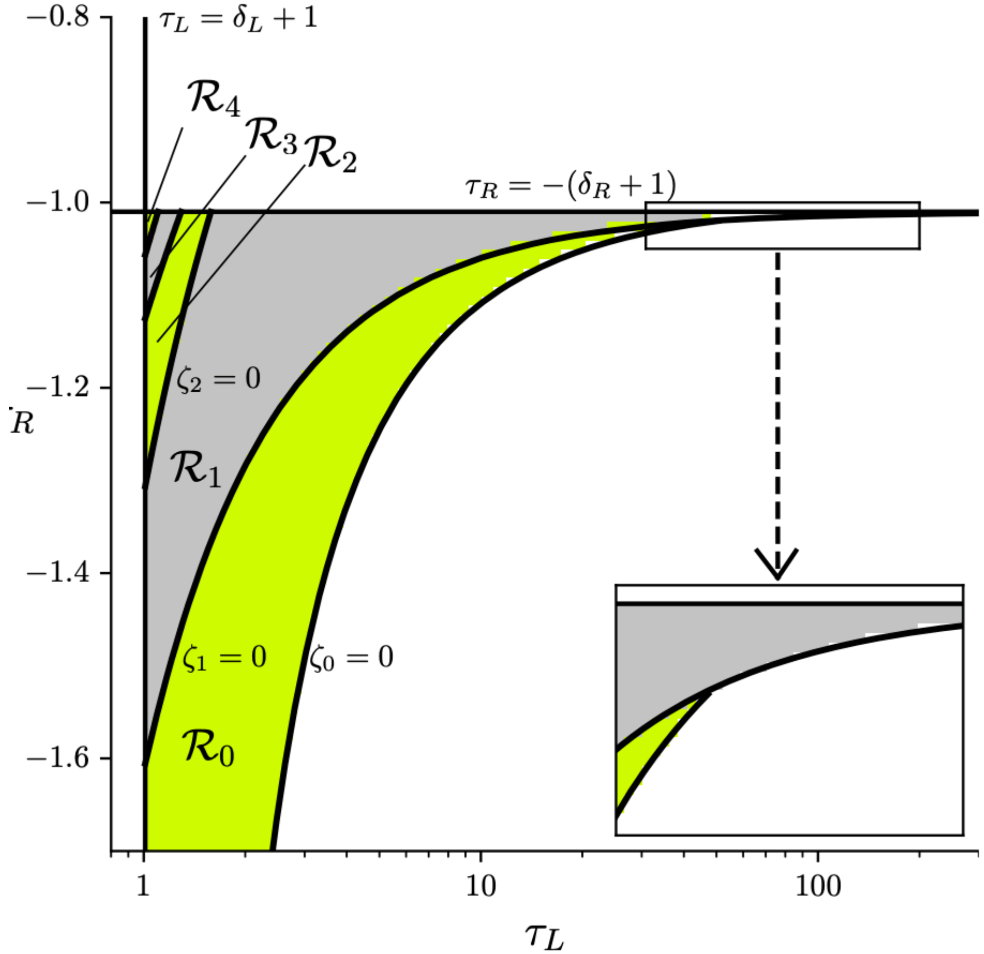

Figure: Sketch of two-dimensional cross-section of Φ_{BYG} when $\delta_L = \delta_R = 0.01$

Results

Theorem (Ghosh & Simpson, 2022)

The $\mathscr{R}_n$ *are non-empty, mutually disjoint, and converge to the fixed point (1, 0, -1, 0) as* $n \to \infty$. *Moreover*

$$\Phi_{\mathrm{BYG}} \subset \bigcup_{n=0}^{\infty} \mathscr{R}_n \,.$$

Let, $\Lambda(\xi) = \mathrm{cl}(W^u(X))$.

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Let, $\Lambda(\xi) = \mathrm{cl}(W^u(X))$.

Theorem (Ghosh & Simpson, 2022)

With any $\xi \in \mathscr{R}_0$, $\Lambda(\xi)$ is bounded, connected, and invariant. Moreover, it is chaotic (positive Lyapunov exponent).

Results

Theorem (Ghosh & Simpson, 2022)

For any $\xi \in \mathscr{R}_n$, *where* $n \geq 0, g^n(\xi) \in \mathscr{R}_0$ *and there exist mutually disjoint sets* $S_0, S_1, \ldots, S_{2^n-1} \subset \mathbb{R}^2$ *such that* $f_\xi(S_i) = S_{(i+1) \mod 2^n}$ *and*

$$f_\xi^{2^n} \mid S_i \text{ is affinely conjugate to } f_{g^n(\xi)} \mid \Lambda(g^n(\xi))$$

for each $i \in \{0,1,\ldots,2^n - 1\}$. *Moreover,*

$$\bigcup_{i=0}^{2^n-1} S_i = \mathrm{cl}\left(W^u(\gamma_n)\right),$$

where γ_n *is a saddle-type periodic solution of our map having the symbolic itinerary* $\mathscr{F}^n(R)$:

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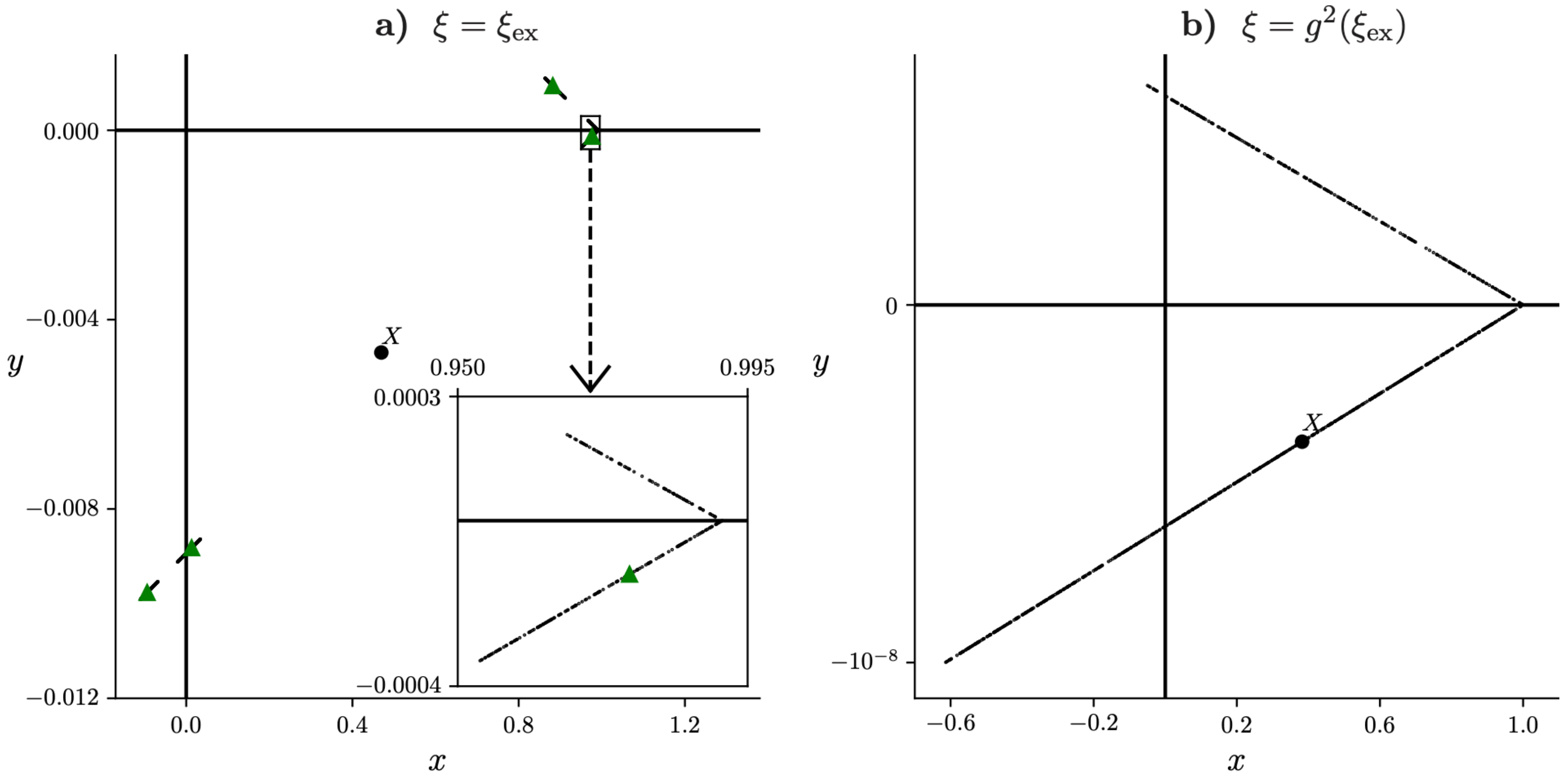
Results

n	$\mathcal{F}^n(R)$
0	R
1	LR
2	RRLR
3	LRLRRRLR
4	RRLRRRLRLRLRRRLR

Table: The first 5 words in the sequence generated by the repeatedly applying the substitution rule $(L, R) \mapsto (RR, LR)$ to $\mathscr{W} = R$.

Results


a) $\xi = \xi_{ex}$
0.000
−0.004
−0.008
−0.012
y
X
0.950
0.995
0.0003
−0.0004
0.0
0.4
0.8
1.2
x
b) $\xi = g^2(\xi_{ex})$
0
y
X
-10^{-8}
−0.6
−0.2
0.2
0.6
1.0
x

Devaney chaos

Theorem (Ghosh & Simpson, 2022)

Let $\xi \in \Phi_{\mathrm{BYG}}$ *and suppose* $J_1(\xi) > 1$ *and* $\lambda_L^s + |\lambda_R^s| < 1$. *Then* $W^s(X)$ *is dense in a triangular region containing* Λ.

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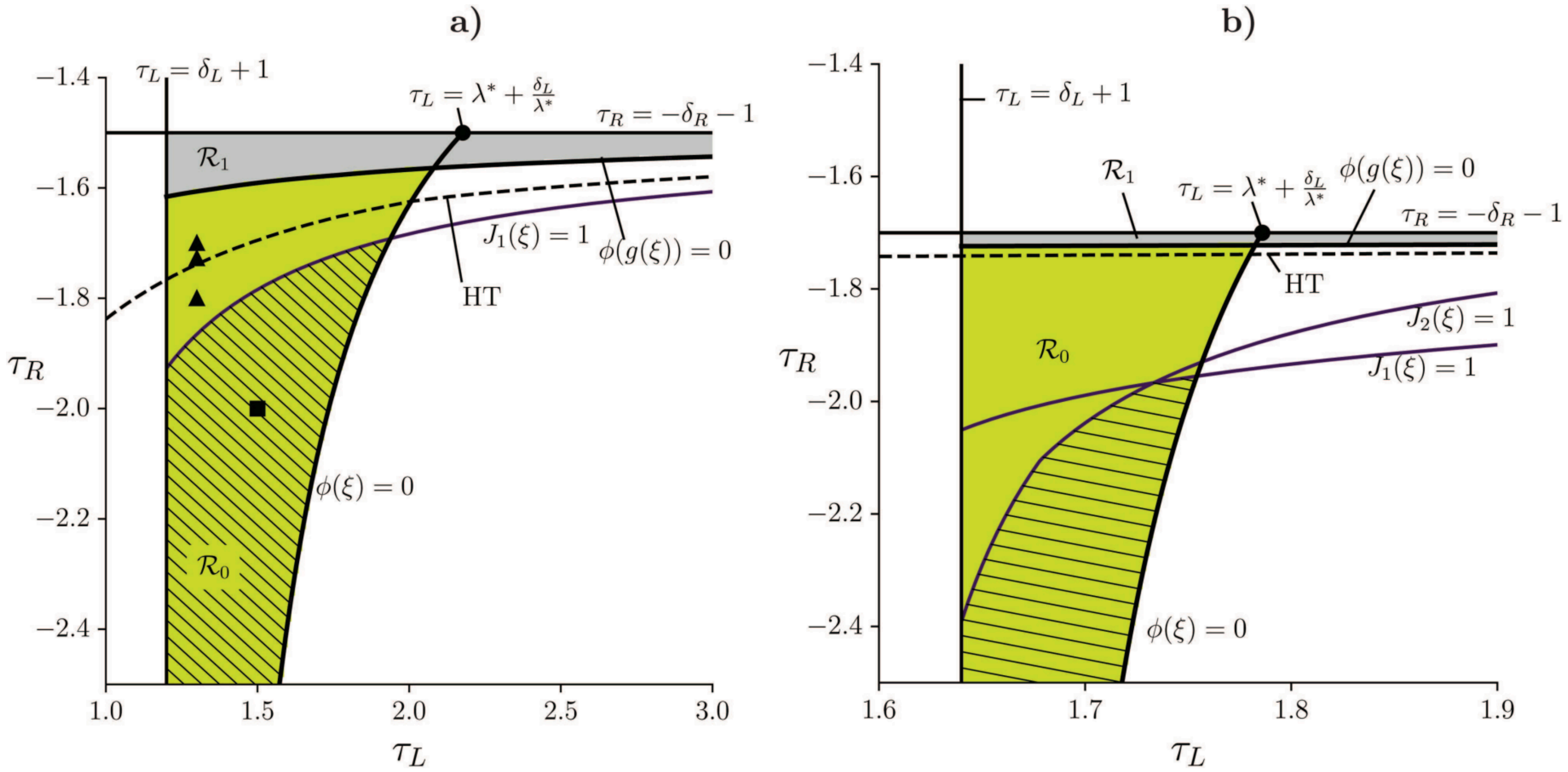
Theorem (Ghosh & Simpson, 2022)

Let $\xi \in \Phi_{\mathrm{BYG}}$ *and suppose* $J_1(\xi) > 1$ *and* $J_2(\xi) < 1$. *Then* f_ξ *is chaotic in the sense of Devaney on* Λ.

Devaney Chaos



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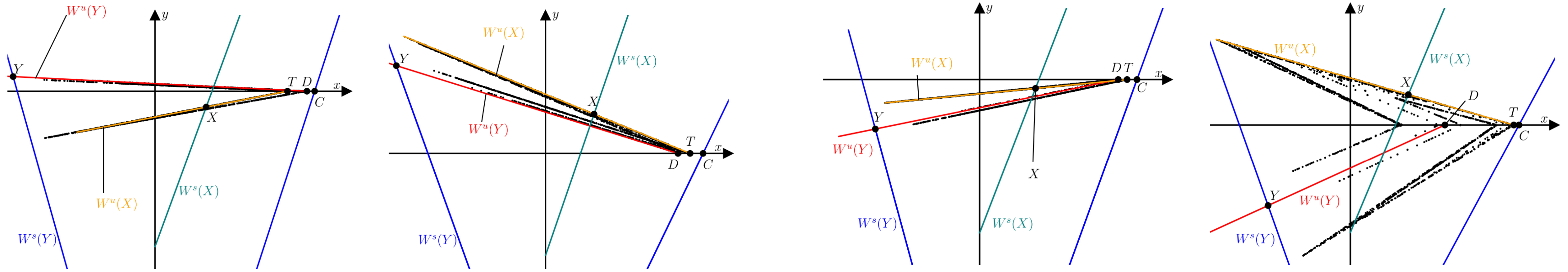
Generalised parameter region

Now we consider the more generalised parameter region considering the orientation-reversing and non-invertible cases,

$$\Phi = \{\, \xi \in \mathbb{R}^4 \mid \tau_L > |\delta_L + 1|,\ \tau_R < -|\delta_R + 1| \,\}.$$



(a) $\delta_L > 0, \delta_R > 0$ (b) $\delta_L > 0, \delta_R < 0$ (c) $\delta_L < 0, \delta_R > 0$ (d) $\delta_L < 0, \delta_R < 0$

Invariant expanding cone

Chaos in Φ_{BYG} can be proved by constructing an invariant expanding cone in tangent space (Glendinning & Simpson, 2021). We have extended this to Φ.

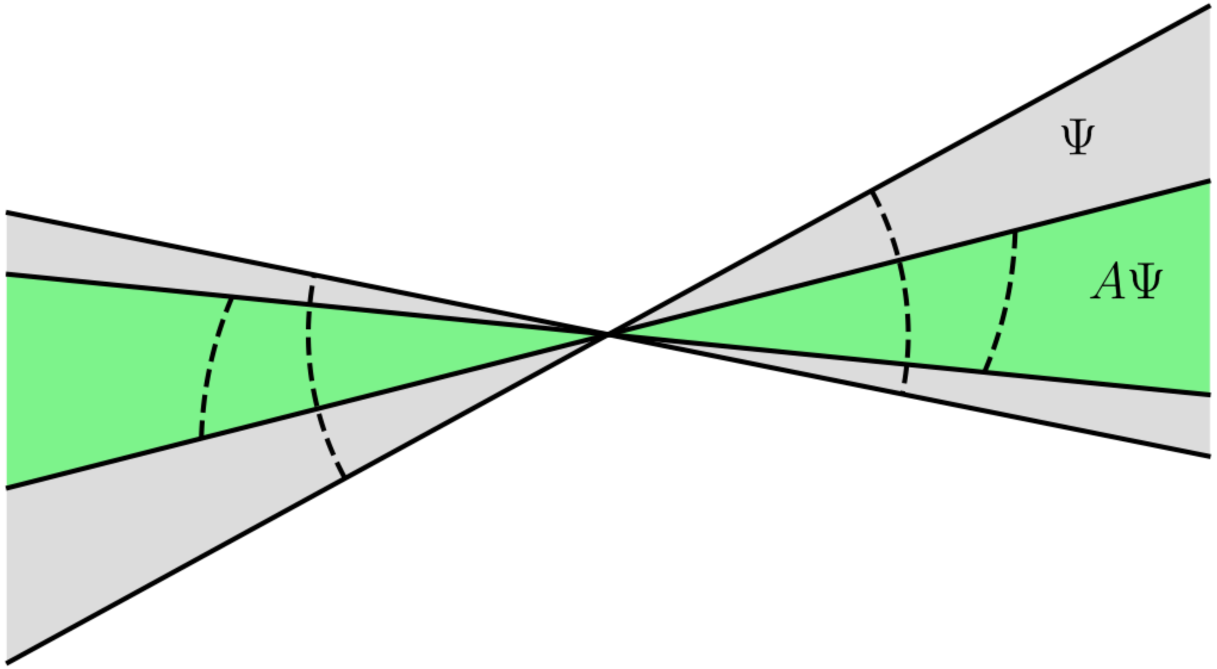


Figure: A sketch of an invariant expanding cone Ψ and its image $A\Psi = \{Av \,|\, v \in \Psi\}$, given $A \in \mathbb{R}^{2\times 2}$.

Robust chaos in a generalised setting

Theorem (Ghosh, McLachlan, & Simpson, 2023)

For any $\xi \in \Phi_{\text{trap}} \cap \Phi_{\text{cone}}$, the normal form has a topological attractor with a positive Lyapunov exponent.

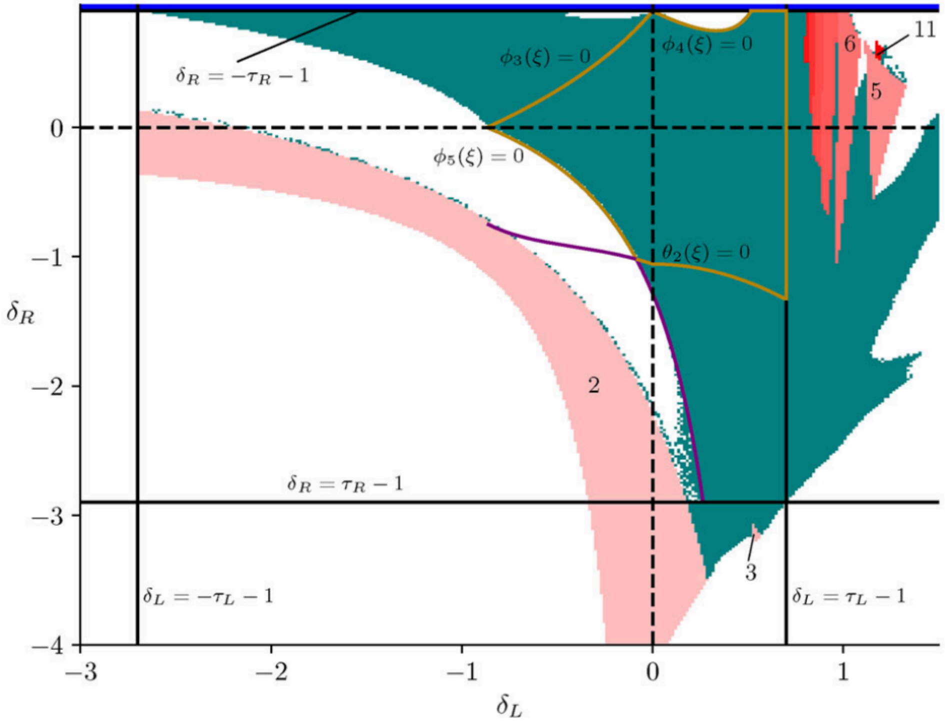


Figure: A 2D sketch of $\Phi_{\text{trap}} \cap \Phi_{\text{cone}} \subset \mathbb{R}^4$

The orientation reversing case

1. Let

$$\Phi^{(2)} = \{\xi \in \Phi \mid \delta_L < 0, \delta_R < 0\}$$

be the subset of Φ for which the BCNF is orientation-reversing.

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   be the subset of Φ for which the BCNF is orientation-reversing.

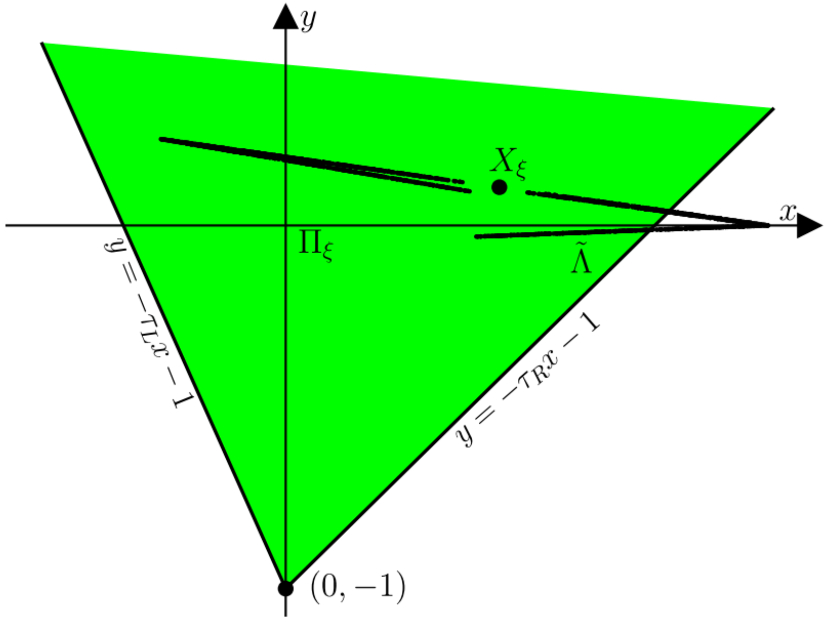
2. The attractor Λ which is again a closure of the unstable manifold of X has a crisis at $\zeta_0^{(2)} = 0$ where

$$\zeta_0^{(2)} = \phi^-(\xi) = \delta_R - \left(\delta_R + \tau_R - (1 + \lambda_R^u)\lambda_L^u\right)\lambda_L^u .$$

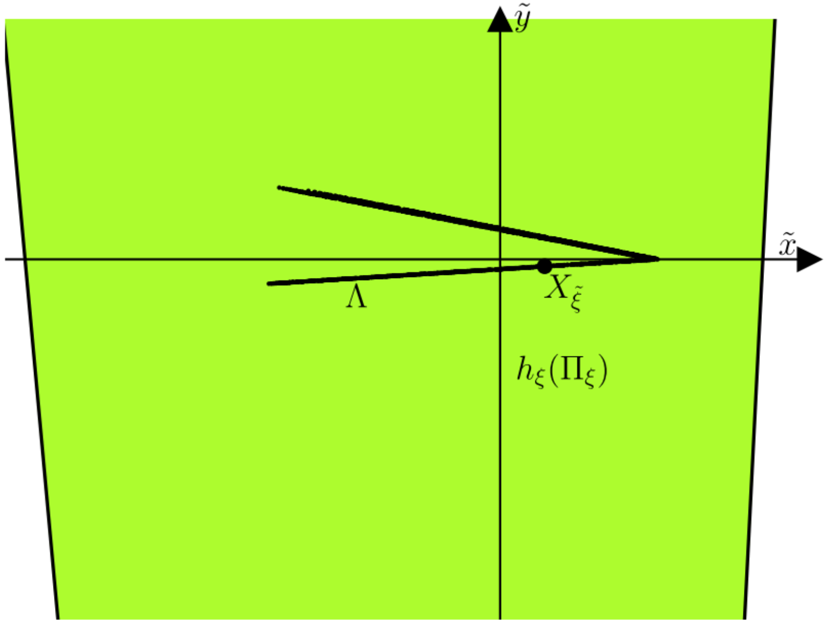
The orientation reversing case

Proposition (Ghosh, McLachlan, & Simpson, 2024)

If $\xi \in \mathscr{R}_n^{(2)}$ *with* $n \geq 1$, *then* $g(\xi) \in \mathscr{R}_{n-1}^{(1)}$.



(a) $\xi = \xi_{\text{ex}}^{(2)} \in \mathcal{R}_1^{(2)}$

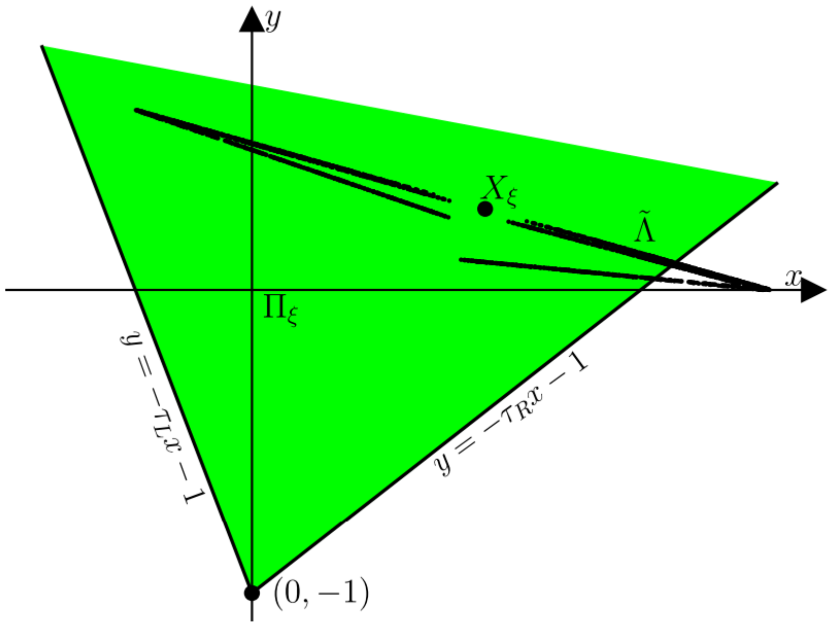


(b) $\xi = g\left(\xi_{\text{ex}}^{(2)}\right) \in \mathcal{R}_0^{(1)}$

The non-invertible case ($\delta_L > 0, \delta_R < 0$)

Proposition (Ghosh, McLachlan, & Simpson, 2024)

If $\xi \in \mathscr{R}_n^{(3)}$ *with* $n \geq 1$*, then* $g(\xi) \in \mathscr{R}_{n-1}^{(3)}$.



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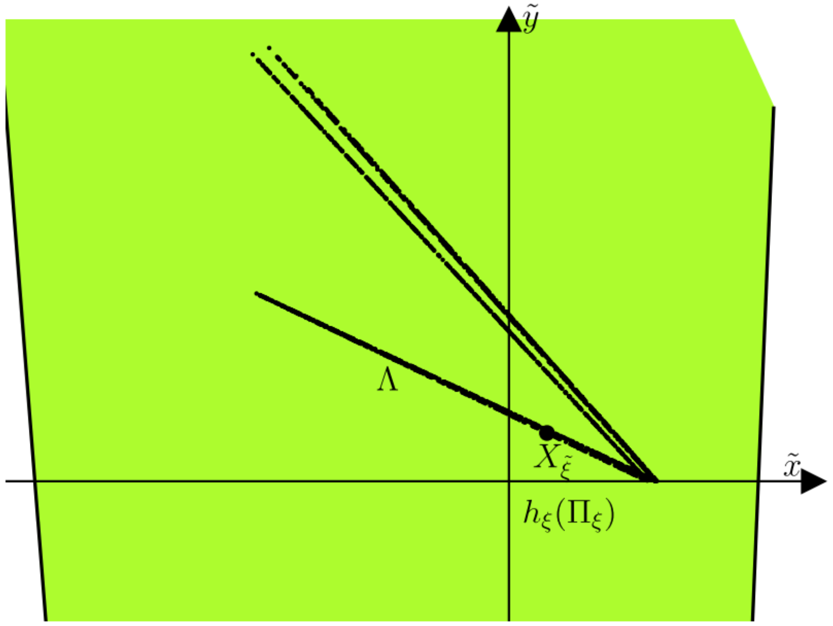

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The non-invertible case ($\delta_L < 0, \delta_R > 0$)

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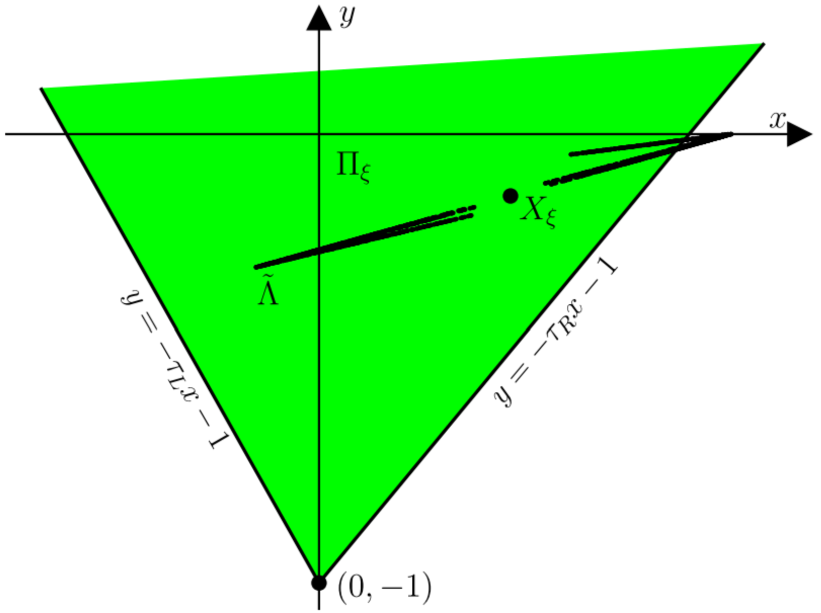
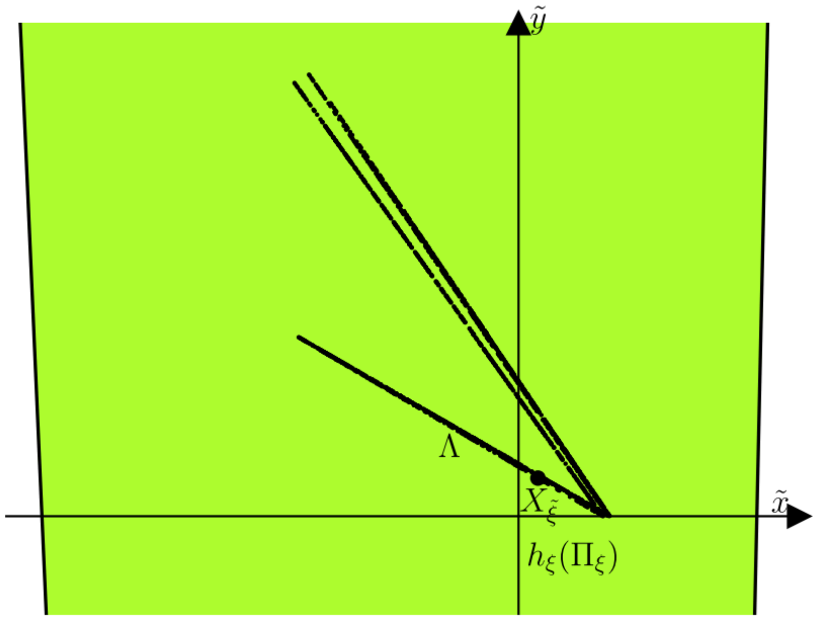


(a) $\xi = \xi_{\text{ex}}^{(4)} \in \mathcal{R}_1^{(4)}$



(b) $\xi = g\left(\xi_{\text{ex}}^{(4)}\right) \in \mathcal{R}_0^{(3)}$

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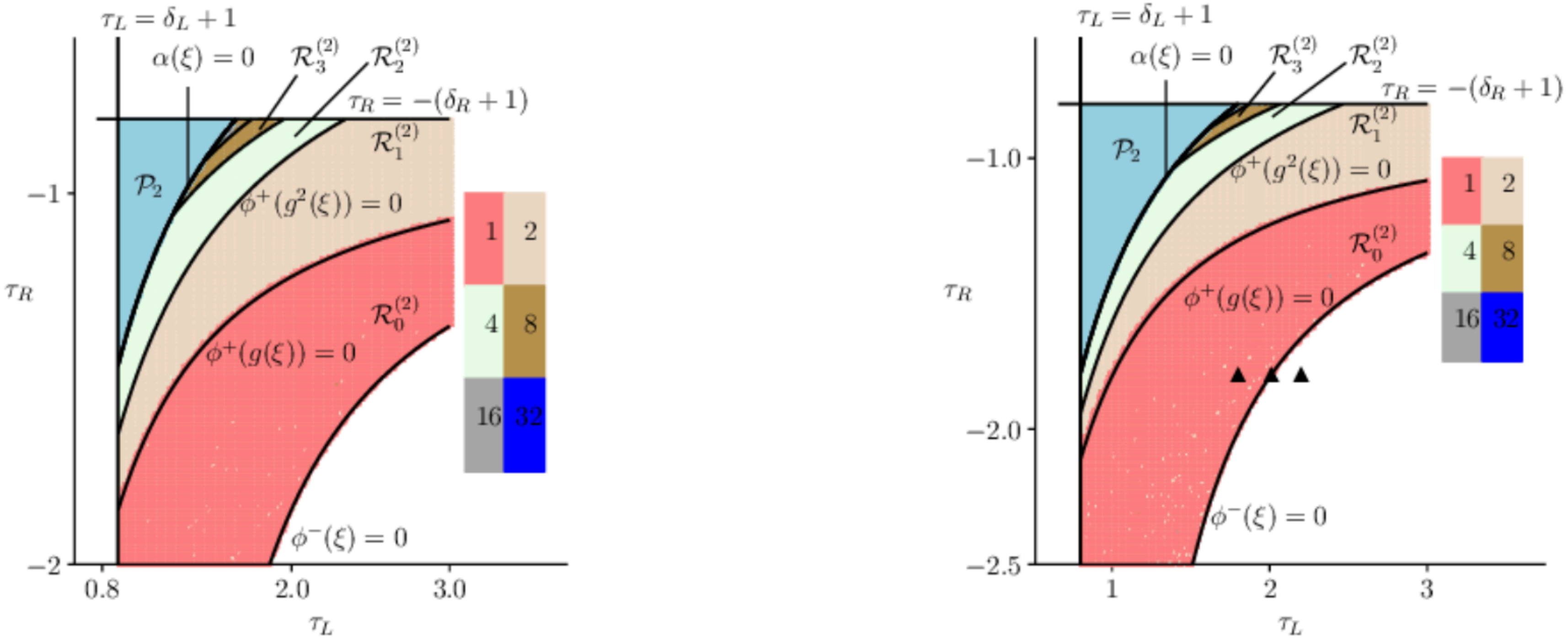
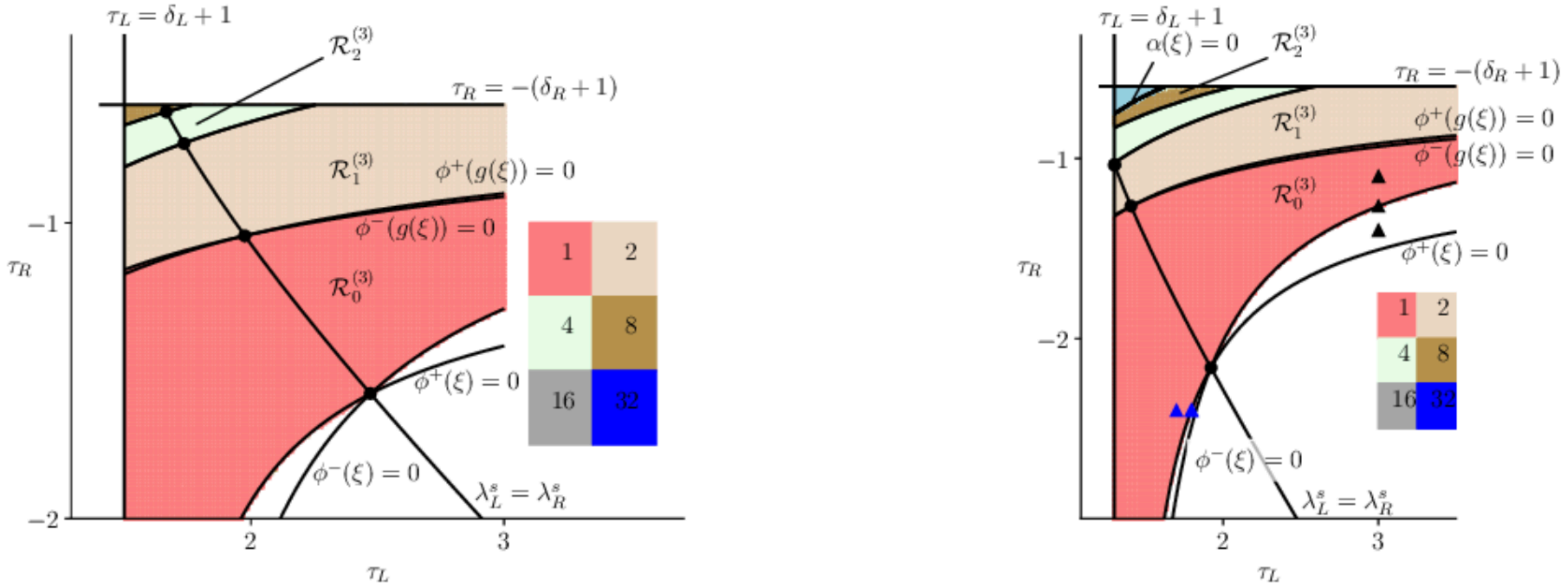
Numerics

We verify these using Eckstein's greatest common divisor algorithm (2007), described by Avrutin *et al.*, 2007 to estimate from sample orbits the number of connected components in the attractor



(a) $\delta_L = -0.1, \delta_R = -0.2.$ (b) $\delta_L = -0.2, \delta_R = -0.2.$

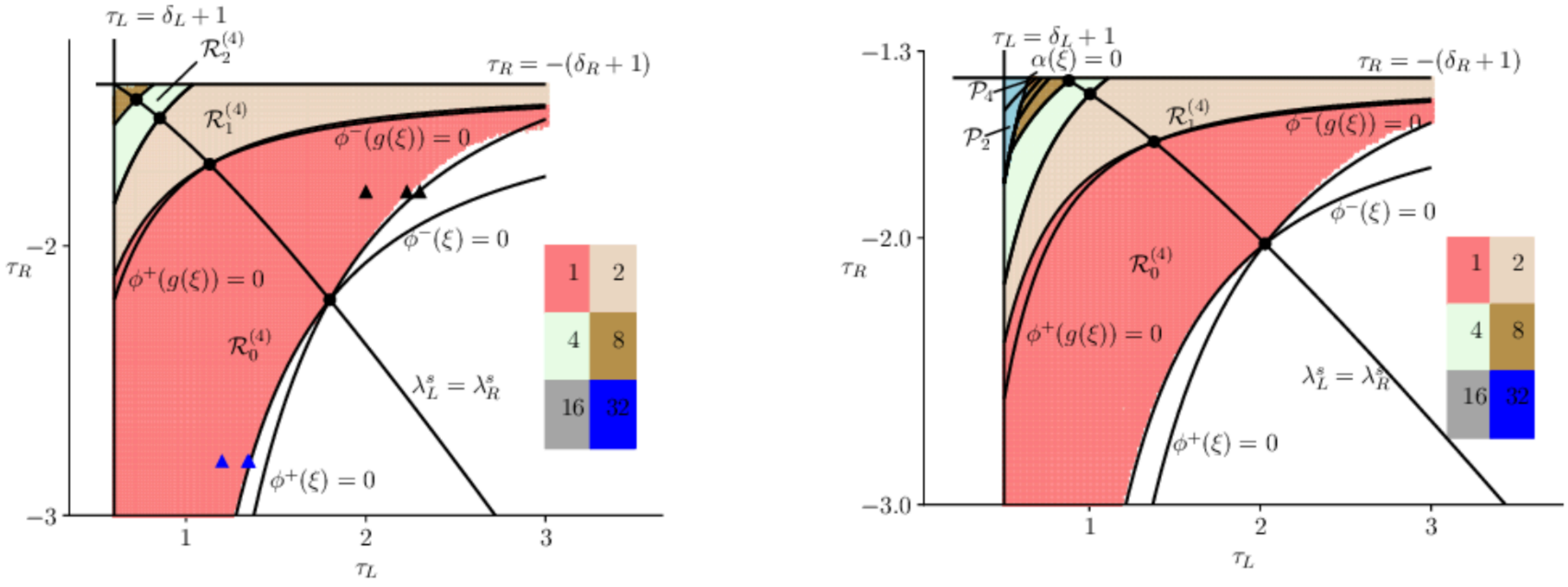
Numerics



(a) $\delta_L = 0.5, \delta_R = -0.4.$

(b) $\delta_L = 0.3, \delta_R = -0.4.$

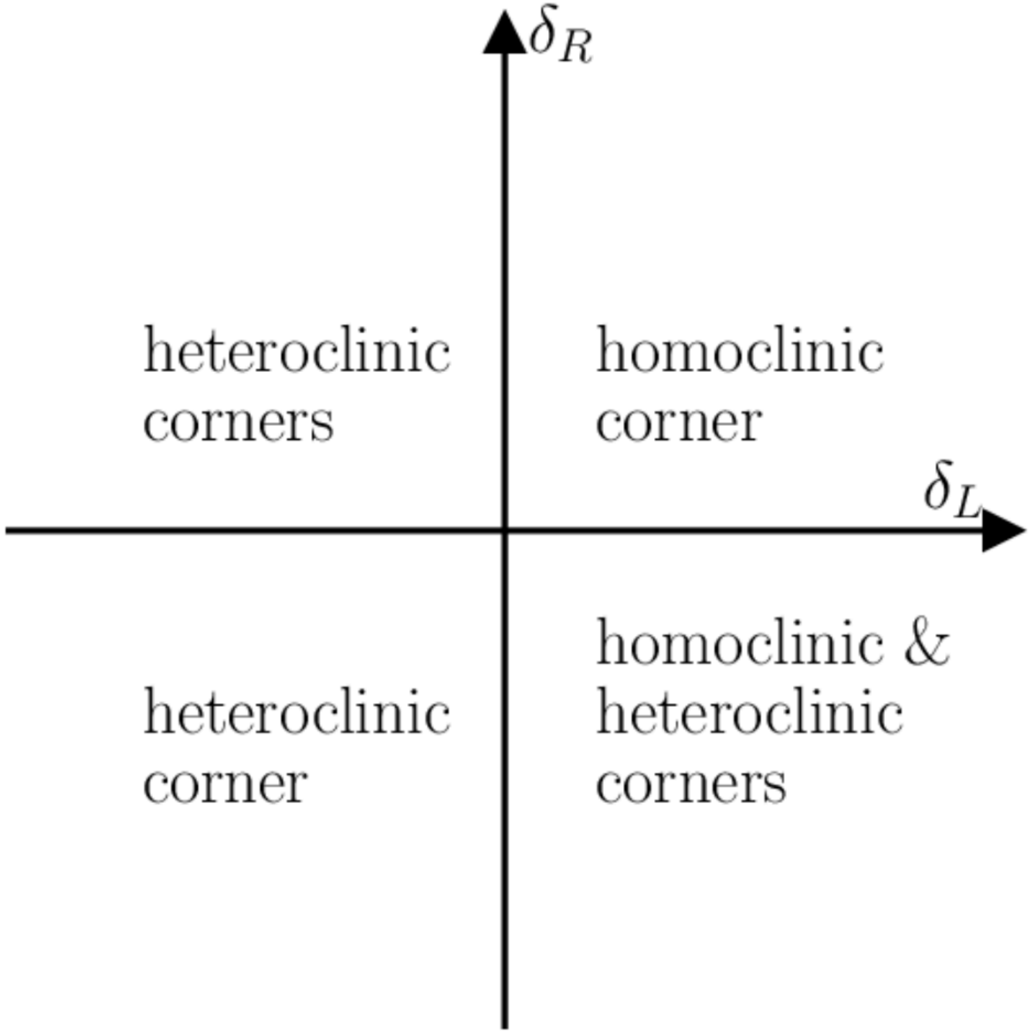
Numerics



(a) $\delta_L = -0.4, \delta_R = 0.4.$

(b) $\delta_L = -0.5, \delta_R = 0.4.$

Numerics



δ_R
δ_L
heteroclinic corners
homoclinic corner
heteroclinic corner
homoclinic & heteroclinic corners

Higher-dimensional setting

Let $n \geq 2$. Suppose $\alpha > 1$ is an eigenvalue of A_L, and $-\beta < -1$of A_R with multiplicity one, and all other eigenvalues of A_L and A_R have modulus at most $0 < r < 1$.

Theorem (Ghosh, & Simpson, 2024)

Holding the above assumption and

$$r(n-1) < \frac{3}{7}\left(1 - \frac{1}{\alpha}\right), \qquad r(n-1) < \frac{3}{7}\left(1 - \frac{1}{\beta}\right), \qquad r(n-1) < \frac{1}{10}\left(\frac{1}{\alpha} + \frac{1}{\beta} - 1\right),$$

Then f has a topological attractor with a positive Lyapunov exponent.

Higher-dimensional setting

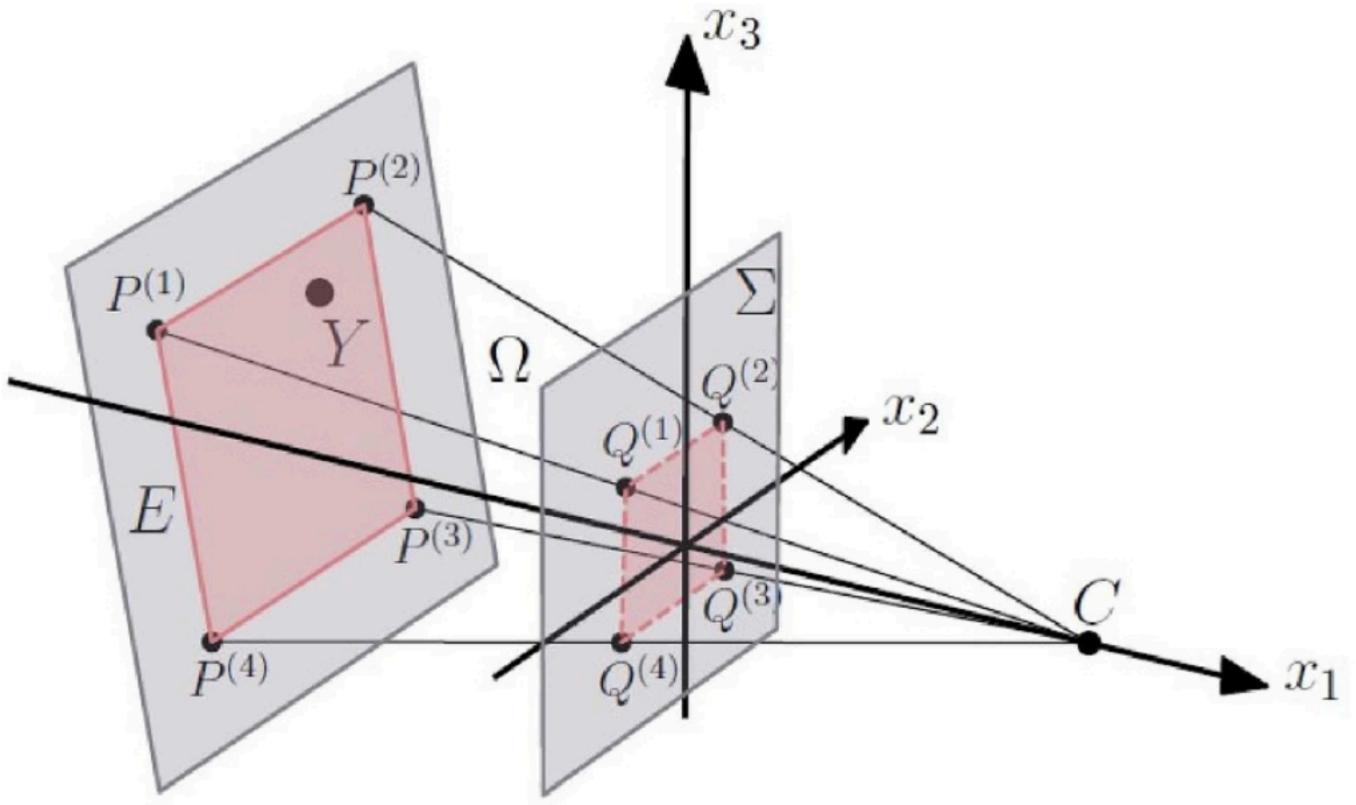


Figure: Construction of a forward invariant region Ω for $n = 3$

Higher-dimensional setting

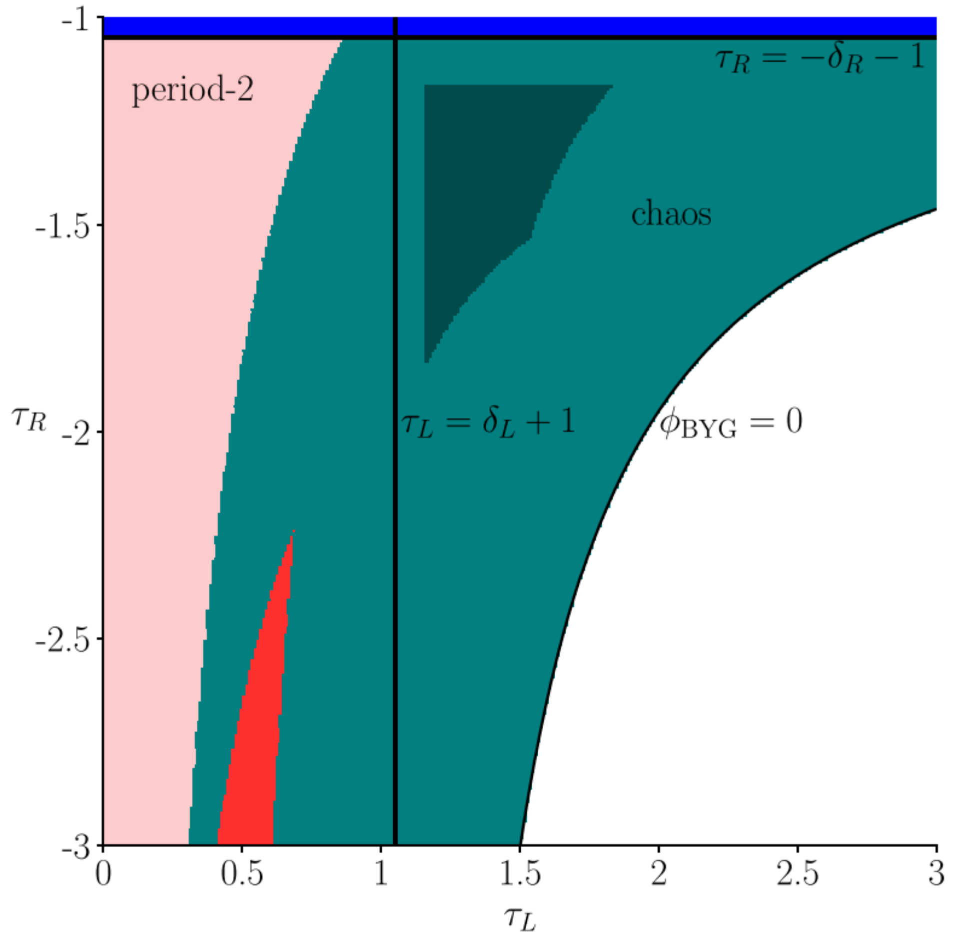


Figure: Robust chaos parameter region for the 2D map. We choose $n = 2$ for simplicity

Future directions

1. We expect our construction of the 2D setting to be adapted to verify robust chaos beyond the boundaries reported.

source: https://looneytunes.fandom.com/wiki/Wile_E._Coyote

Future directions

1. We expect our construction of the 2D setting to be adapted to verify robust chaos beyond the boundaries reported.

2. Renormalisation schemes based on other substitution rules to explain parameter regimes where the BCNF has attractors with three connected components, for example.

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3. Maps with multiple directions of instability is just as relevant: “wild chaos”.

4. Application: higher-dimensional construction as the key space for an encryption scheme.

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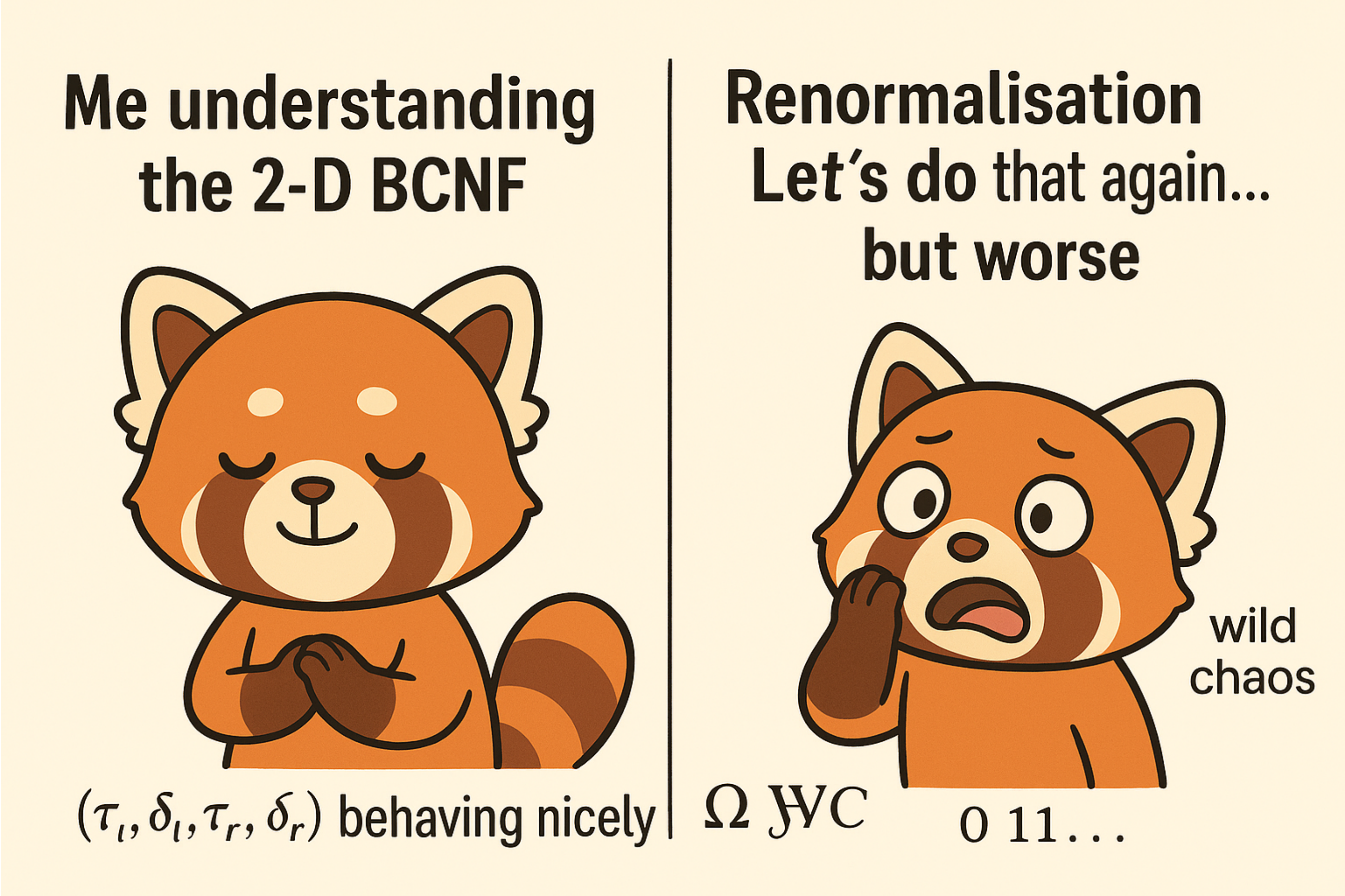
Questions, comments, suggestions?



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