Bifurcation structure of robust chaos in 2D piecewise-linear maps

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Introduction

Piecewise-linear maps are used for modeling systems with switches, thresholds and other abrupt events. The two dimensional *border-collision normal form* that we study is given by

$$f_\xi(x,y) = \begin{cases} \begin{bmatrix} \tau_L & 1 \\ -\delta_L & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}, & x \le 0, \\ \begin{bmatrix} \tau_R & 1 \\ -\delta_R & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}, & x \ge 0, \end{cases}$$

with $(x,y) \in \mathbb{R}^2$, and $\xi = (\tau_L, \delta_L, \tau_R, \delta_R) \in \mathbb{R}^4$ are the parameters.

Renormalization operator

Although the second iterate f_ξ^2 has four pieces, relevant dynamics arise in only two of these. We have

$$f_\xi^2(x,y) = \begin{cases} \begin{bmatrix} \tau_L\tau_R - \delta_L & \tau_R \\ -\delta_R\tau_L & -\delta_R \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} \tau_R + 1 \\ -\delta_R \end{bmatrix}, & x \le 0, \\ \begin{bmatrix} \tau_R^2 - \delta_R & \tau_R \\ -\delta_R\tau_R & -\delta_R \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} \tau_R + 1 \\ -\delta_R \end{bmatrix}, & x \ge 0. \end{cases}$$

Now f_ξ^2 can be transformed to $f_{g(\xi)}$, where g is the *renormalisation operator* $g : \mathbb{R}^4 \to \mathbb{R}^4$, given by

$$\begin{aligned} \tilde\tau_L &= \tau_R^2 - 2\delta_R, \\ \tilde\delta_L &= \delta_R^2, \\ \tilde\tau_R &= \tau_L\tau_R - \delta_L - \delta_R, \\ \tilde\delta_R &= \delta_L\delta_R. \end{aligned}$$

Results I

We consider the parameter region

$$\Phi = \left\{ \xi \in \mathbb{R}^4 \middle| \tau_L > \delta_L + 1, \delta_L > 0, \tau_R < -(\delta_R + 1), \delta_R > 0 \right\},$$

where

$$\Phi_{\rm BYG} = \{ \xi \in \Phi | \phi_4(\xi) > 0 \}.$$

- Theorem 1: The $\mathcal{R}_n$ are non-empty, mutually disjoint, and converge to the fixed point $(1,0,-1,0)$ as $n \to \infty$. Moreover,
$$\Phi_{\rm BYG} \subset \bigcup_{n=0}^{\infty} \mathcal{R}_n.$$
- Theorem 2: For the map f_ξ with any $\xi \in \mathcal{R}_0$, the attractor $\Lambda(\xi)$ is bounded, connected, and invariant. Moreover, it is chaotic (positive Lyapunov exponent).
- Theorem 3: For any $\xi \in \mathcal{R}_n$ where $n \ge 0$, $g^n(\xi) \in \mathcal{R}_0$ and there exist mutually disjoint sets $S_0, S_1, \ldots, S_{2^n-1} \subset \mathbb{R}^2$ such that $f_\xi(S_i) = S_{(i+1) \bmod 2^n}$ and
$$f_\xi^{2^n}|_{S_i} \text{ is affinely conjugate to } f_{g^n(\xi)}|_{\Lambda(g^n(\xi))}$$
for each $i \in \{0,1,\ldots,2^n-1\}$. Moreover,
$$\bigcup_{i=0}^{2^n-1} S_i = {\rm cl}(W^u(\gamma_n)),$$
where γ_n is a saddle-type periodic solution of our map f_ξ having the symbolic itinerary $\mathcal{F}^n(R)$ given by the substitution rule $(L,R) \mapsto (RR, LR)$ to $\mathcal{W} = R$.

Results II

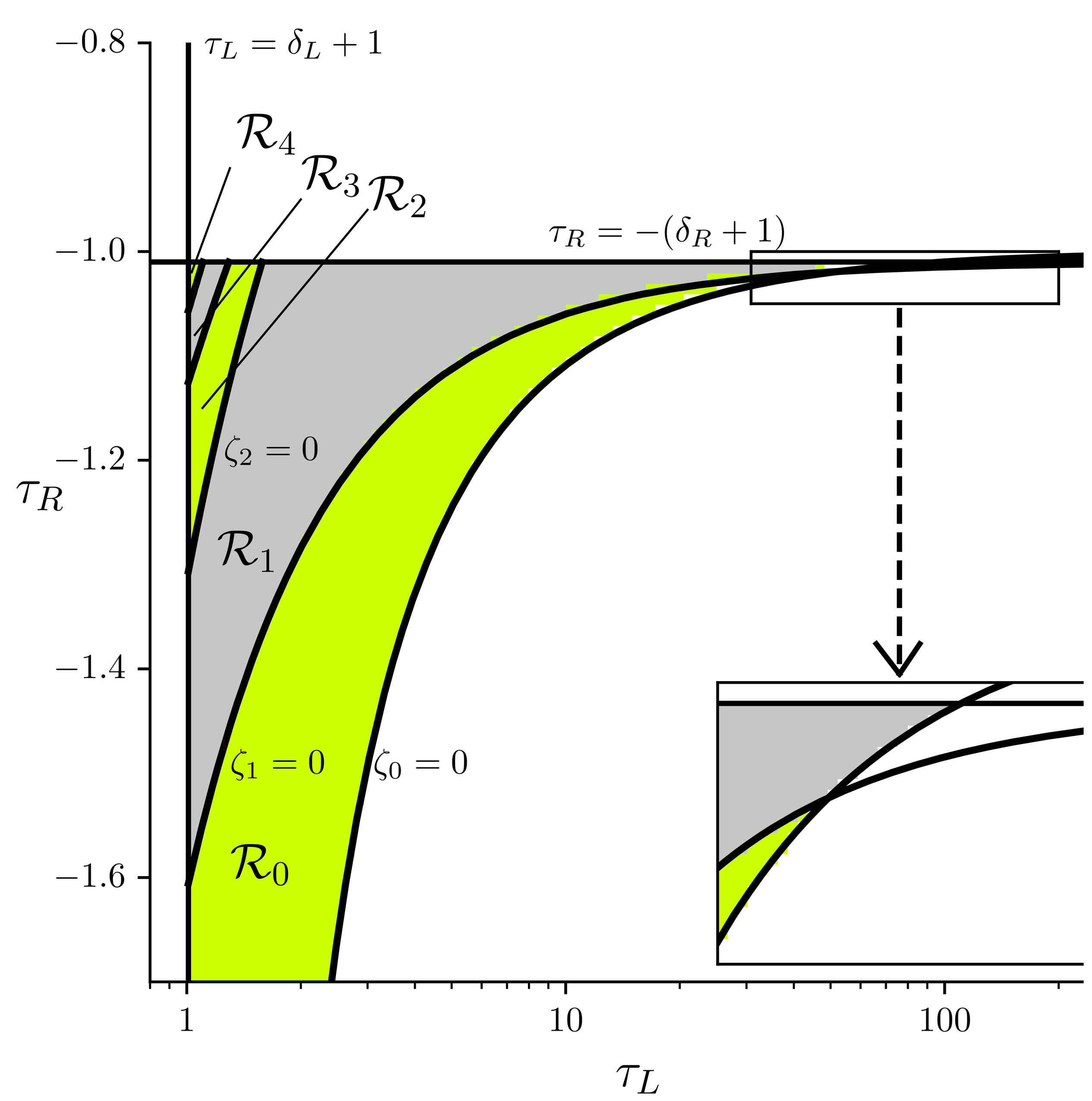

Fig. 1: Sketch of parameter region $\Phi_{\rm BYG}$ with $\delta_L > 0, \delta_R > 0$.

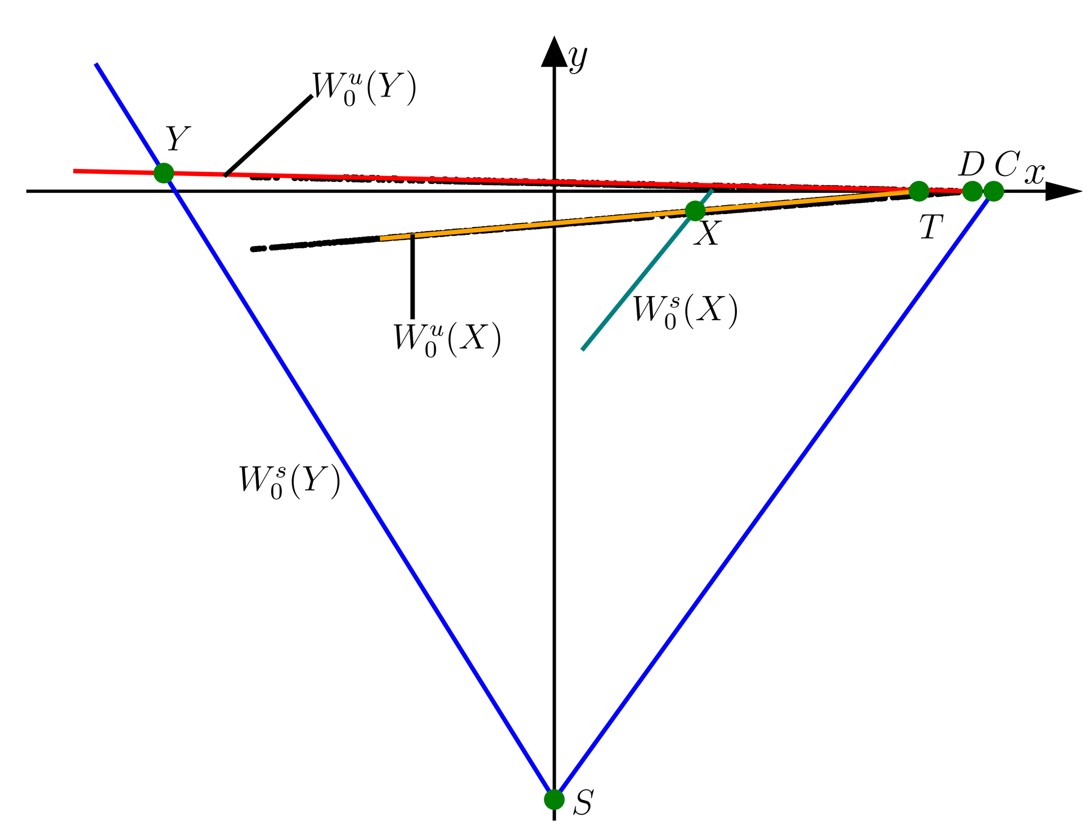

Fig. 2: Sketch of the phase portrait of f_ξ with $\xi \in \Phi_{\rm BYG}$.

Next, we consider the more generalised parameter region considering the orientation-reversing and non-invertible cases,

$$\Phi = \left\{ \xi \in \mathbb{R}^4 \;\middle|\; -\tau_L - 1 < \delta_L < \tau_L - 1, \tau_R - 1 < \delta_R < -\tau_R - 1 \right\}, \tag{1}$$

where we define

$$\Phi_{\rm trap} = \{ \xi \in \Phi | \; \phi_i(\xi) > 0, i = 1, \ldots, 5 \}, \tag{2}$$

and

$$\Phi_{\rm cone} = \{ \xi \in \Phi | \theta_i(\xi) \ge 0, i = 1, \ldots, 7 \}. \tag{3}$$

- Theorem 4: Suppose $\xi \in \Phi_{\rm trap} \cap \Phi_{\rm cone}$, then f_ξ has a topological attractor with the property that it is chaotic in sense of positive maximal Lyapunov exponent on each point on the attractor.

Results III

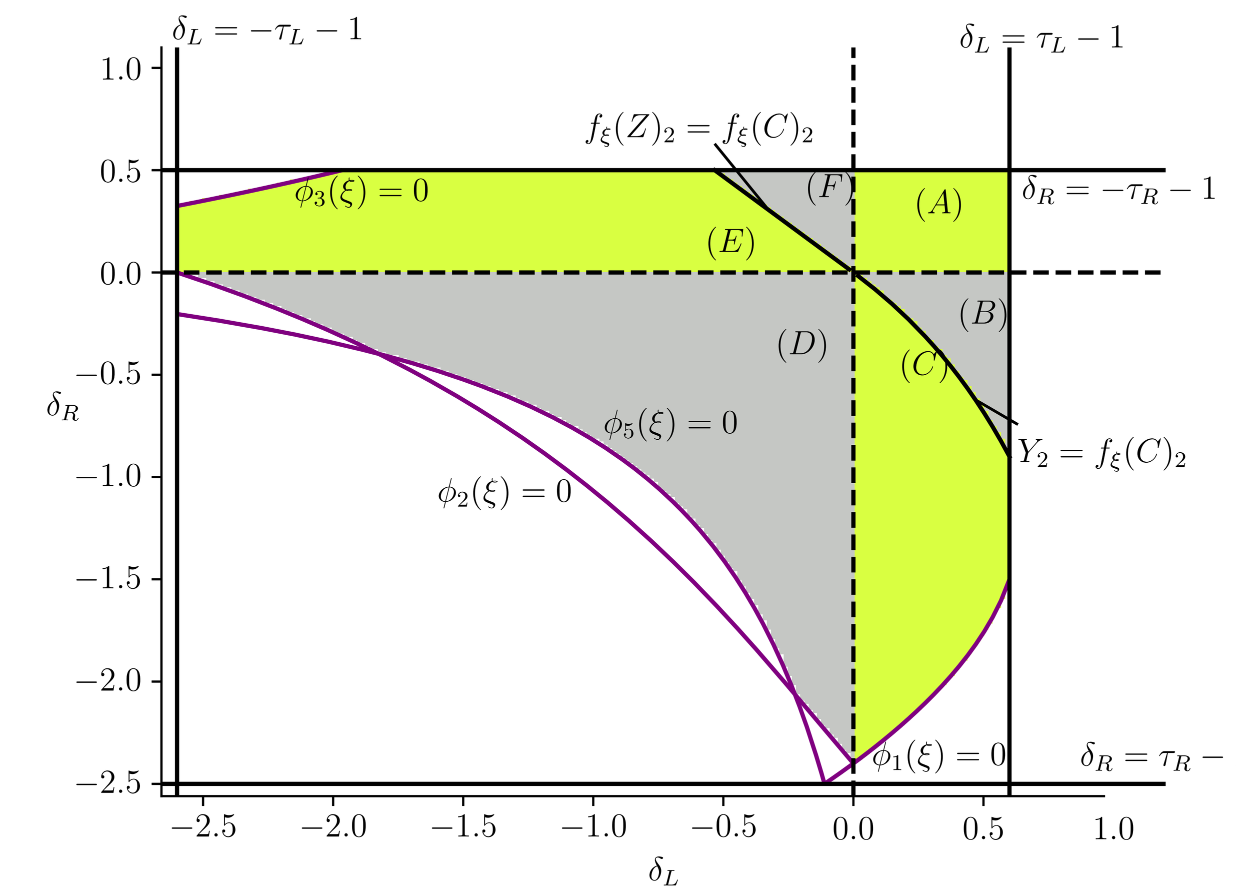

Fig. 3: A 2D slice of $\Phi_{\rm trap}$.

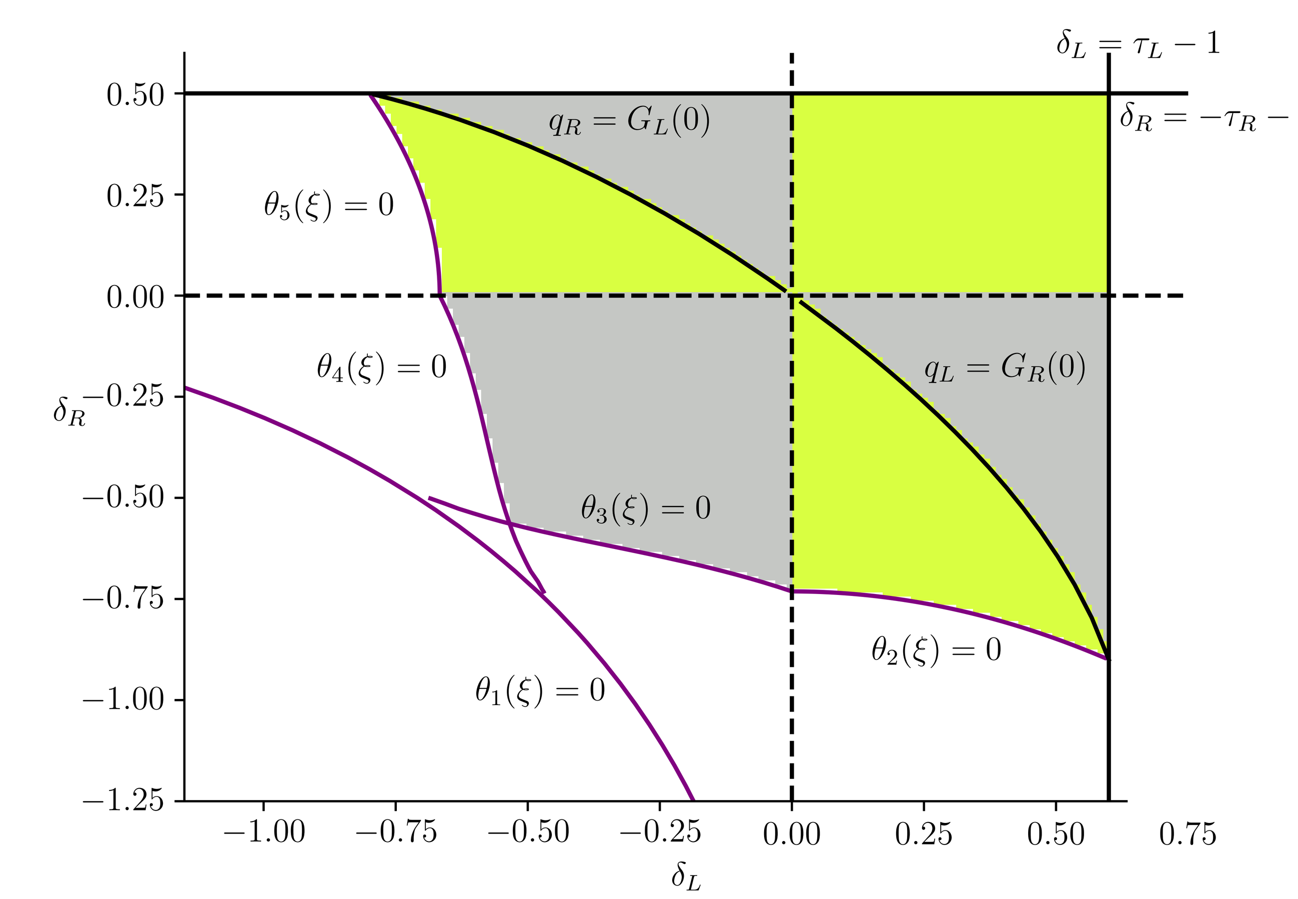

Fig. 4: A 2D slice of $\Phi_{\rm cone}$.

Acknowledgements

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