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Dynamical Properties of Denatured Morris-Lecar Neurons

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Collaborators

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AUT
TE WĀNANGA ARONUI O TĀMAKI MAKAU RAU

(b) Sishu S. Muni

DIGITAL UNIVERSITY KERALA

A denatured Morris-Lecar neuron model

- A simplified variant of the Morris-Lecar neuron was introduced in their book by Schaeffer and Cain, which has been dubbed as the *denatured* Morris-Lecar (dML) model.

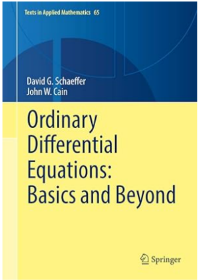

Figure: Book by Schaeffer and Cain[1].

[1] D. Schaeffer and J. Cain, "Ordinary differential equations: Basics and beyond". (Springer, 2018).

A denatured Morris-Lecar neuron model

- The model equations are

$$\dot{x} = x^2(1-x) - y + I,$$
$$\dot{y} = Ae^{\alpha x} - \gamma y.$$

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- The exponential term in y models a negative feedback, corresponding to the dynamics of the refractory period.

A denatured Morris-Lecar neuron model

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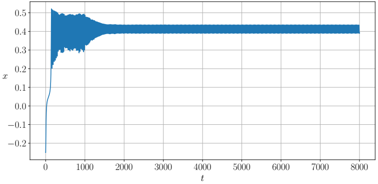
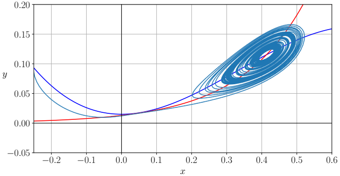
- Whereas α is a control parameter influencing the exponential growth rate of y.



(a) Time series



(b) Phase portrait

A denatured Morris-Lecar neuron model

- The dML model is closely comparable to a FitzHugh-Nagumo type neuron model which can be written as

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$$\dot{x} = x^2(1 - x) - y + I,$$
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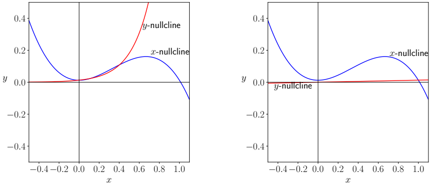
- Both models have the same x-nullclines with differing y-nullclines. The y-nullclines curve upward pertaining to the exponential growth term $Ae^{\alpha x}$, whereas for FHN the y-nullclines are straight lines pertaining to the linear term Ax.



(a) dML (b) FHN

Figure: For parameter values $A = 0.0041$, $\alpha = 5.276$, $\gamma = 0.315$, and $I = 0.012347$.

Qualitative analysis

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- The equilibrium can be computed from the transcendental equations[2]

$$x^2(1-x) - y + I = 0,$$
$$Ae^{\alpha x} - \gamma y = 0,$$

by solving for x.

[2] I. Ghosh, H.O. Fatoyinbo. "I. Ghosh, H.O. Fatoyinbo. "Fractional order induced bifurcations in Caputo-type denatured Morris-Lecar neurons." in Commun. Nonlinear Sci. Numer. Simul. (2025): 108984.

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$$x^2(1-x) - y + I = 0,$$
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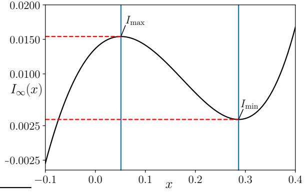
  by solving for x.
- We can define:

$$I_\infty(x) = \frac{A}{\gamma}e^{\alpha x} - x^2(1-x).$$



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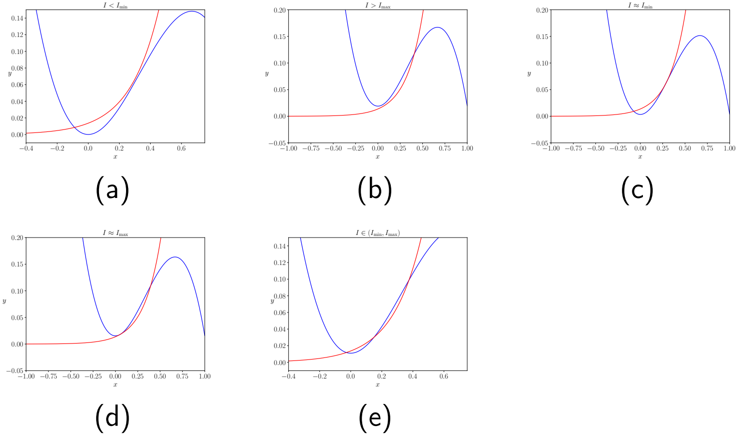

Numerical Bifurcation Analysis

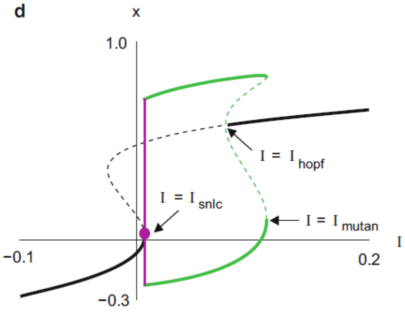

Figure: (a) SNLC: Saddle Node Limit Cycle, (b) I_{mutan}: a mutual annihilation bifurcation occurs at $I = I_{\mathrm{mutan}}$. See D. Schaeffer and J. Cain,(Springer, 2018).

Numerical Bifurcation Analysis

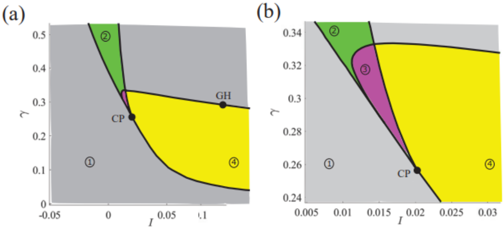

Figure: A codimension-two bifurcation diagram of the dML model in the (I, γ)-plane[3].

[3] H.O. Fatoyinbo, *et al.* "Numerical bifurcation analysis of improved denatured morris-lecar neuron model". In 2022 international conference on decision aid sciences and applications (DASA) (pp. 55-60). IEEE (2022).

A slow-fast variant

- The slow-fast version of the dML also introduced by Schaeffer and Cain is given by

$$\dot{x} = x^2(1 - x) - y + I,$$
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- where

$$I'(x) = \frac{1}{60}\left[1 + \tanh\left(\frac{0.05 - x}{0.001}\right)\right]$$

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  is the smoothed-out version of a step function.
- the parameter ε is a small perturbation parameter that separates the time scales and is sometimes referred to as the *time-scale parameter*.

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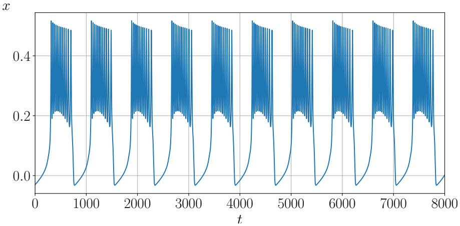

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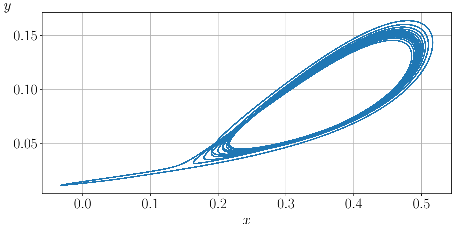

(b) Phase portrait

Figure: We observe a periodic bursting behavior. Here $A = 0.0041$, $\alpha = 5.276$, $\gamma = 0.315$, and $\varepsilon = 0.001$. The initial condition $x(0)$ is sampled uniformly from the range $[-1, 1]$. Furthermore $(y(0), I(0)) = (0.1, 0.012347)$.

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- Bistability leads to bursting: vary I slowly in time.

[4] E. Izhikevich, "Dynamical systems in neuroscience". (MIT press, 2007).

A slow-fast variant

- Bistability leads to bursting: vary I slowly in time.
- This kind of bursting is classified as *fold/homoclinic* type[4] where the transition from the resting state to the spiking limit cycle occurs via a saddle-node (fold) bifurcation and from the spiking state to the resting state via a saddle homoclinic orbit bifurcation.

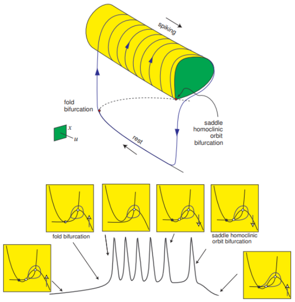

Figure 9.25: "Fold/homoclinic" bursting. The resting state disappears via saddle-node (fold) bifurcation, and the spiking limit cycle disappears via saddle homoclinic orbit bifurcation.

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Two-coupled dML neurons

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- Two connected neurons can be mathematically modeled using a directional coupling strategy.

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Two-coupled dML neurons

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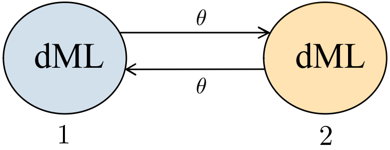
- In our work[5] a gap-junction coupling replicating a bidirectional electrical synapse is utilized. The neurons are considered identical.



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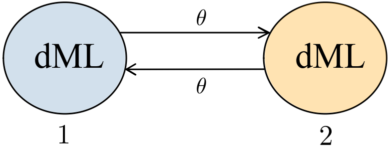

- The model equations are

$$\dot{x}_1 = x_1^2(1 - x_1) - y_1 + I_1 + \theta(x_2 - x_1), \quad \dot{y}_1 = Ae^{\alpha x_1} - \gamma y_1, \quad \dot{I}_1 = \varepsilon(I'(x_1) - I_1),$$
$$\dot{x}_2 = x_2^2(1 - x_2) - y_2 + I_2 + \theta(x_1 - x_2), \quad \dot{y}_2 = Ae^{\alpha x_2} - \gamma y_2, \quad \dot{I}_2 = \varepsilon(I'(x_2) - I_2).$$

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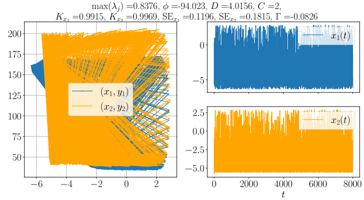
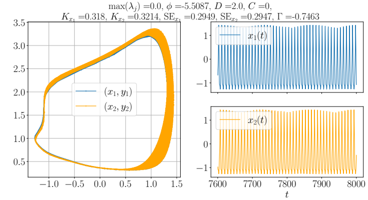
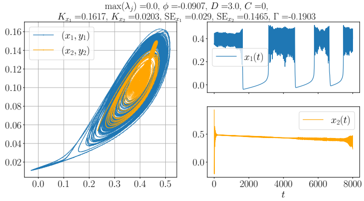
Time series & phase portraits



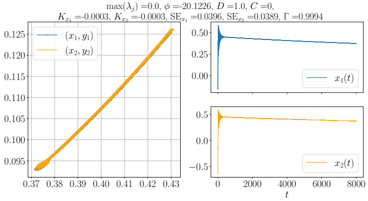
(a) $\theta = -15$, $\varepsilon = 0.0002$: Hyperchaotic



(b) $\theta = -1$, $\varepsilon = 0.0002$: Quasiperiodic



(c) $\theta = 0$, $\varepsilon = 0.0002$: irregular bursting



(d) $\theta = 10$, $\varepsilon = 0.0002$: Decay oscillations

Codimension-one bifurcation diagram

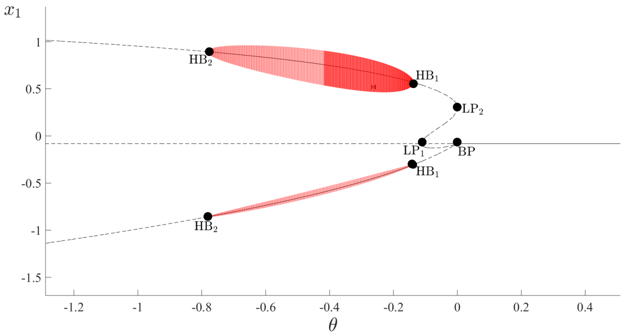


Figure: Codimension-one bifurcation diagram of the coupled fast subsystem. Solid [dashed] curves correspond to stable [unstable] solutions and red curves are limit cycles. HB, LP, and BP represent Hopf bifurcation, saddle-node bifurcation of an equilibrium and branch point respectively.

The $0-1$ test for detecting chaos[6]

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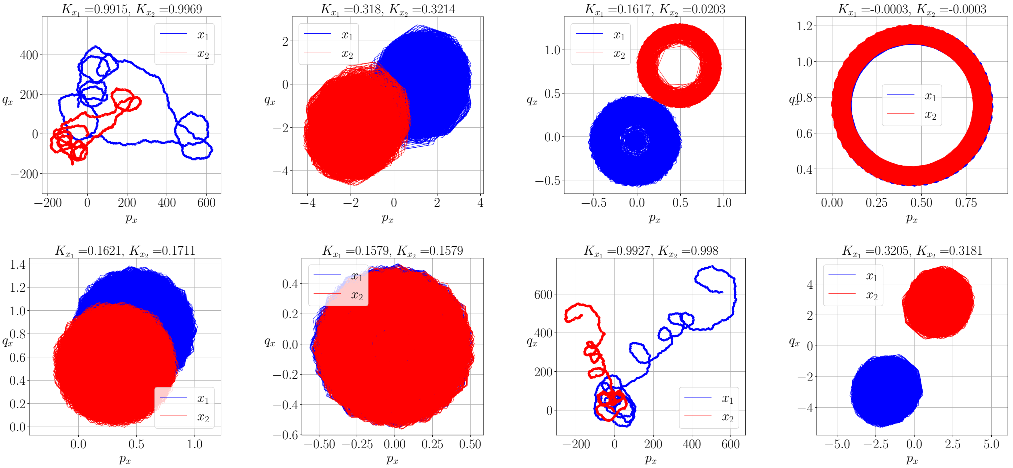

[6]G. Gottwald and I. Melbourne, On the implementation of the 0–1 test for chaos, SIAM J. Appl. Dyn. Syst. 8, 129 (2009).

Cross-correlation coefficient for synchronization

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- The cross-correlation coefficient between node 1 and 2 given by

$$\Gamma = \frac{\langle \tilde{x}_1(t)\tilde{x}_2(t)\rangle}{\sqrt{\langle \tilde{x}_1(t)^2\rangle\langle \tilde{x}_2(t)^2\rangle}},$$

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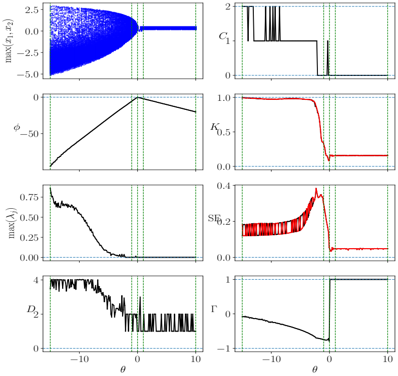
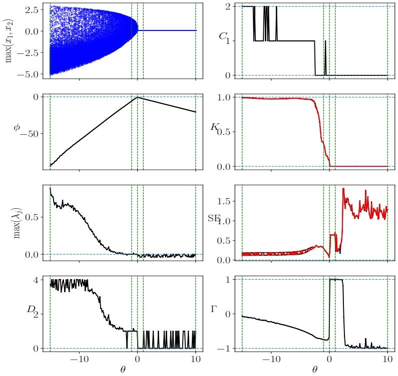
- When $\Gamma = 1$ it means both the nodes are in phase and completely synchronized, whereas $\Gamma = -1$ represents anti-phase synchrony.

Numerics



(a) $\varepsilon = 0.001$



(b) $\varepsilon = 0.1$

Fractional order version

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$$

- Here C stands for "Caputo" and $\beta \in (0, 1]$ is the order of the integral, also known as the *memory index*.

Qualitative analysis

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Theorem

Suppose

i) $x^*(2-3x^*)-\gamma > 0$, *and*

ii) $-\gamma x^*(2-3x^*)+\alpha A e^{\alpha x^*} < 2\sqrt{-\gamma - x^*(2-3x^*)}\cos(\frac{\beta\pi}{2})$.

Then an equilibrium point (x^*, y^*) *of the fractional order system is asymptotically stable.*

Theorem

Suppose $I \in (I_{\min}, I_{\max})$. *Then this branch of equilibrium points is completely unstable.*

- From the above theorem we can directly see that $\delta(x^*) < 0$ implies one of the two eigenvalues is positive and the other negative, meaning the equilibrium point on this branch is a saddle, irrespective of the fractional order $\beta \in (0, 1]$.

Theorem

Suppose $I = I_{\min}$ *or* $I = I_{\max}$. *Then the fractional order system has a* saddle-node *bifurcation.*

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Theorem

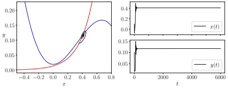
Suppose $I < I_{\min}$ *or* $I > I_{\max}$. *Then*

i) *the stability of an equilibrium point of the system depends on the sign of* $\tau(x^*)$,

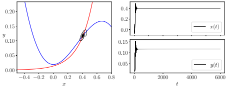
ii) *for* $\tau(x^*) \geq 0$ *the equilibrium is asymptotically stable if and only if the order*

$$\beta < \beta^* = \frac{2}{\pi}\cos^{-1}\left(\min\left(1, \frac{-\gamma + x^*(2-3x^*)}{2\sqrt{\alpha A e^{\alpha x^*} - \gamma x^*(2-3x^*)}}\right)\right).$$

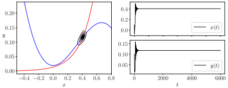
Phase portraits



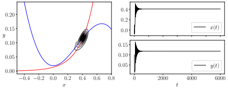
(a) $\beta = 0.9$



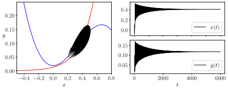
(b) $\beta = 0.92$



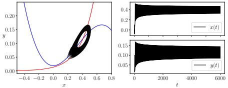
(c) $\beta = 0.94$



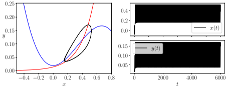
(d) $\beta = 0.96$



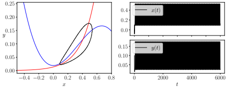
(e) $\beta = 0.98$



(f) $\beta = \beta^*$

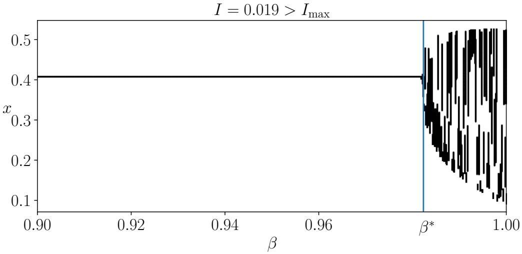


(g) $\beta = 0.99$

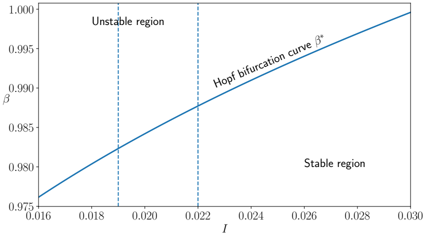


(h) $\beta = 1$

A crude bifurcation diagram



(a) $\beta^* \approx 0.98233$



(b) Hopf bifurcation curve

Discussion/Future work

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- We also aim to study pattern formation in a diffusively coupled chain of neurons
- Delay-induced coupling is an interesting avenue to explore.

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- It would be intriguing to investigate the dynamical behaviour of the coupled neurons as a game-theoretic model.

The End

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Thank you! Questions?