Understanding the Topology of Chaotic Attractors for Piecewise-Linear Maps using Renormalisation.

Indranil Ghosh, Robert I. McLachlan, David J. W. Simpson

School of Mathematical and Computational Sciences
Massey University, Palmerston North, New Zealand

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Border-collision normal form

- Piecewise-linear maps arise when modeling systems with switches, thresholds and other abrupt events.
- In our project, we study the two-dimensional *border-collision normal form* (Nusse & Yorke, 1992), given by

$$f_\xi(x,y) = \begin{cases} \begin{bmatrix} \tau_L & 1 \\ -\delta_L & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}, & x \leq 0, \\ \begin{bmatrix} \tau_R & 1 \\ -\delta_R & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}, & x \geq 0. \end{cases}$$

- Here $(x,y) \in \mathbb{R}^2$, and $\xi = (\tau_L, \delta_L, \tau_R, \delta_R) \in \mathbb{R}^4$ are the parameters.

Phase portrait of a chaotic attractor

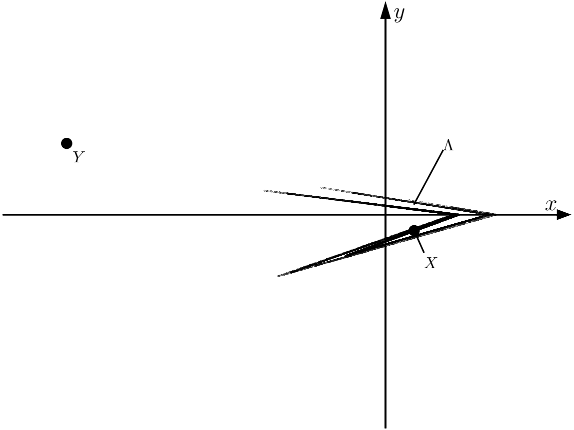


Figure: A sketch of the phase portrait of f_ξ with $\xi \in \Phi_{\mathrm{BYG}}$.

Renormalisation operator

- Renormalisation involves showing that, for some member of a family of maps, a higher iterate or induced map is conjugate to different member of this family of maps.
- Although the second iterate f_ξ^2 has four pieces, relevant dynamics arise in only two of these. We have

$$f_\xi^2(x,y) = \begin{cases} \begin{bmatrix} \tau_L\tau_R - \delta_L & \tau_R \\ -\delta_R\tau_L & -\delta_R \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} \tau_R + 1 \\ -\delta_R \end{bmatrix}, & x \le 0, \\ \begin{bmatrix} \tau_R^2 - \delta_R & \tau_R \\ -\delta_R\tau_R & -\delta_R \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} \tau_R + 1 \\ -\delta_R \end{bmatrix}, & x \ge 0. \end{cases}$$

Renormalisation operator

- Now f_ξ^2 can be transformed to $f_{g(\xi)}$, where g is the *renormalisation operator* (Ghosh & Simpson, 2022.) $g : \mathbb{R}^4 \to \mathbb{R}^4$, given by

$$\begin{aligned}\tilde{\tau}_L &= \tau_R^2 - 2\delta_R,\\ \tilde{\delta}_L &= \delta_R^2,\\ \tilde{\tau}_R &= \tau_L\tau_R - \delta_L - \delta_R,\\ \tilde{\delta}_R &= \delta_L\delta_R.\end{aligned}$$

- We perform a coordinate change to put f_ξ^2 in the normal form :

$$\begin{bmatrix}\tilde{x}'\\ \tilde{y}'\end{bmatrix} = \begin{cases}\begin{bmatrix}\tilde{\tau}_L & 1\\ -\tilde{\delta}_L & 0\end{bmatrix}\begin{bmatrix}\tilde{x}\\ \tilde{y}\end{bmatrix} + \begin{bmatrix}1\\ 0\end{bmatrix}, & \tilde{x} \le 0,\\ \begin{bmatrix}\tilde{\tau}_R & 1\\ -\tilde{\delta}_R & 0\end{bmatrix}\begin{bmatrix}\tilde{x}\\ \tilde{y}\end{bmatrix} + \begin{bmatrix}1\\ 0\end{bmatrix}, & \tilde{x} \ge 0.\end{cases}$$

Results

- We consider the parameter region

$$\Phi = \left\{ \xi \in \mathbb{R}^4 \middle| \tau_L > \delta_L + 1, \delta_L > 0, \tau_R < -(\delta_R + 1), \delta_R > 0 \right\}.$$

- The stable and the unstable manifolds of the fixed point Y intersect if and only if $\phi^+(\xi) \leq 0$.
- The attractor is often destroyed at $\phi^+(\xi) = 0$ which is a homoclinic bifurcation (Banerjee, Yorke & Grebogi, 1998), and thus focused their attention on the region

$$\Phi_{\mathrm{BYG}} = \left\{ \xi \in \Phi \middle| \phi^+(\xi) > 0 \right\}.$$

where

$$\phi^+(\xi) = \zeta_0 = \delta_R - (\tau_R + \delta_L + \delta_R - (1 + \tau_R)\lambda_L^u)\lambda_L^u.$$

Results

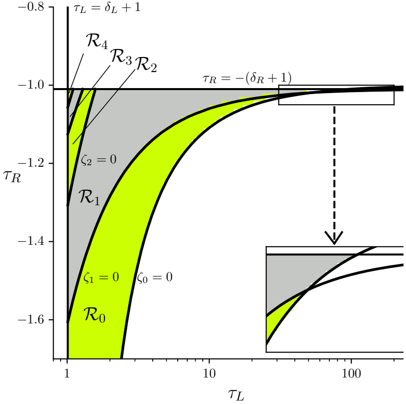


Figure: The sketch of two-dimensional cross-section of Φ_{BYG} when $\delta_L = \delta_R = 0.01$.

Results

Theorem (Ghosh & Simpson, 2022)

The $\mathcal{R}_n$ are non-empty, mutually disjoint, and converge to the fixed point $(1, 0, -1, 0)$ as $n \to \infty$. Moreover,

$$\Phi_{\mathrm{BYG}} \subset \bigcup_{n=0}^{\infty} \mathcal{R}_n.$$

- Let,

$$\Lambda(\xi) = \mathrm{cl}(W^u(X)).$$

Theorem (Ghosh & Simpson, 2022)

For the map f_ξ with any $\xi \in \mathcal{R}_0$, $\Lambda(\xi)$ is bounded, connected, and invariant. Moreover, $\Lambda(\xi)$ is chaotic (positive Lyapunov exponent).

Results

Theorem (Ghosh & Simpson, 2022)

For any $\xi \in \mathcal{R}_n$ *where* $n \geq 0$, $g^n(\xi) \in \mathcal{R}_0$ *and there exist mutually disjoint sets* $S_0, S_1, \ldots, S_{2^n-1} \subset \mathbb{R}^2$ *such that* $f_\xi(S_i) = S_{(i+1) \bmod 2^n}$ *and*

$$f_\xi^{2^n}|_{S_i} \text{ is affinely conjugate to } f_{g^n(\xi)}|_{\Lambda(g^n(\xi))}$$

for each $i \in \{0, 1, \ldots, 2^n - 1\}$. *Moreover,*

$$\bigcup_{i=0}^{2^n-1} S_i = \mathrm{cl}(W^u(\gamma_n)),$$

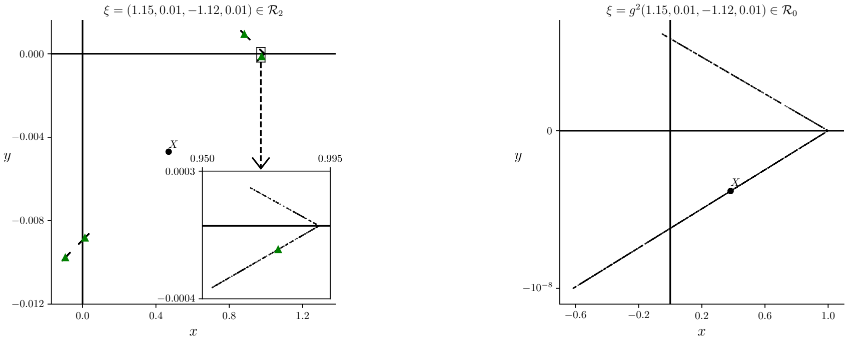
where γ_n *is a saddle-type periodic solution of our map* f_ξ *having the symbolic itinerary* $\mathcal{F}^n(R)$ *given by Table 1.*

Results

n	$\mathcal{F}^n(\mathcal{W})$
0	R
1	LR
2	$RRLR$
3	$LRLRRRLR$
4	$RRLRRRLRLRLRRRLR$

Table: The first 5 words in the sequence generated by repeatedly applying the substitution rule $(L, R) \mapsto (RR, LR)$ to $\mathcal{W} = R$.

Results



$\xi = (1.15, 0.01, -1.12, 0.01) \in \mathcal{R}_2$
0.000
−0.004
−0.008
−0.012
y
0.0
0.4
0.8
1.2
x
X
0.950
0.995
0.0003
−0.0004
$\xi = g^2(1.15, 0.01, -1.12, 0.01) \in \mathcal{R}_0$
0
-10^{-8}
y
−0.6
−0.2
0.2
0.6
1.0
x
X

Generalised parameter region

Now we consider the more generalised parameter region considering the orientation-reversing and non-invertible cases,

$$\Phi = \left\{ \xi \in \mathbb{R}^4 \;\middle|\; \tau_L > |\delta_L + 1|, \tau_R < -|\delta_R + 1| \right\}.$$

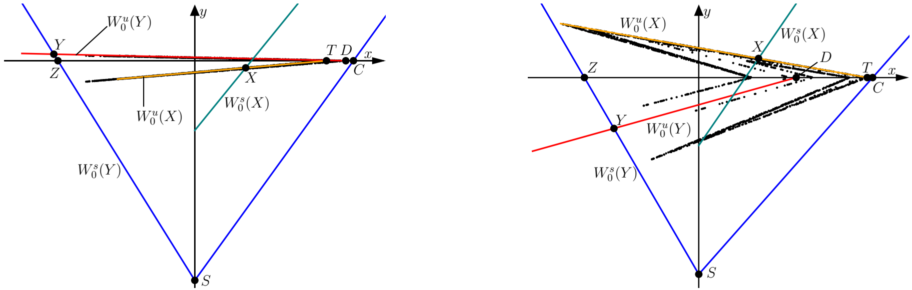
where we define

$$\Phi_{\mathrm{trap}} = \{\xi \in \Phi |\ \phi_i(\xi) > 0, i = 1, \ldots, 5\},$$

and

$$\Phi_{\mathrm{cone}} = \{\xi \in \Phi | \theta_i(\xi) \geq 0, i = 1, \ldots, 3\}.$$

Typical phase portraits

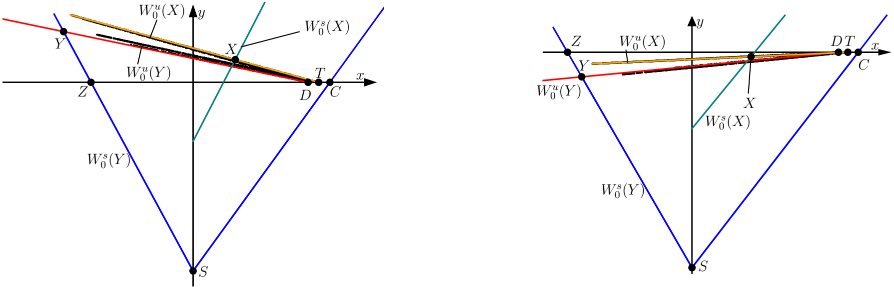


(a) $\delta_L > 0, \delta_R > 0$

(b) $\delta_L < 0, \delta_R < 0$

Figure: Typical phase portraits of the chaotic attractor for the invertible case ($\delta_L\delta_R > 0$).

Typical phase portraits



(a) $\delta_L > 0, \delta_R < 0$

(b) $\delta_L < 0, \delta_R > 0$

Figure: Typical phase portraits of the chaotic attractor for the non-invertible case ($\delta_L \delta_R < 0$).

Invariant expanding cones

Chaos in Φ_{BYG} can be proved by constructing an invariant expanding cone in tangent space (Glendinning & Simpson, 2021). We have extended this to Φ.

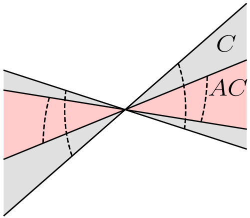


Figure: A sketch of an invariant expanding cone C and its image $AC = \{Av | v \in C\}$, given $A \in \mathbb{R}^{2\times 2}$.

Results

Theorem (Ghosh, McLachlan, & Simpson, 2023)

For any $\xi \in \Phi_{\mathrm{trap}} \cap \Phi_{\mathrm{cone}}$, the normal form f_ξ has a topological attractor with a positive Lyapunov exponent.

Results

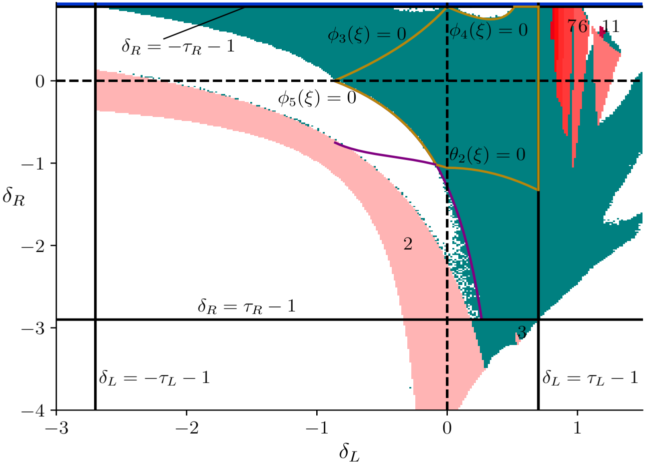


Figure: A 2D slice of $\Phi_{\text{trap}} \cap \Phi_{\text{cone}} \subset \mathbb{R}^4$.

A component computing Algorithm

- This method originates with (Eckstein, 2006) and is described by (Avrutin *et al*, 2007). The effectiveness of the method relies on the following result:

Lemma

Suppose a compact invariant set Ψ *of a continuous map* f *has* $k \geq 2$ *connected components, and* f *has an orbit that visits all components. Then the components can be labelled as* $\Psi_1, \Psi_2, \ldots, \Psi_k$ *such that* $f(\Psi_1) = \Psi_2, f(\Psi_2) = \Psi_3, \ldots, f(\Psi_{k-1}) = \Psi_k$, *and* $f(\Psi_k) = \Psi_1$.

- Now fix ξ and suppose f_ξ has an attractor Λ with $k \geq 1$ connected components. We use the algorithm to calculate k.
- Fix $\varepsilon > 0$ (for example $\varepsilon = 0.001$), $M > 0$ (we used $M = 10^6$), and let $J = \varnothing$.
- Choose some initial point assumed to be in the basin of attraction of Λ and iterate it under f_ξ a reasonably large number of times (we used 10^4 iterations) to remove transient dynamics and obtain a point in Λ, or extremely close to Λ, call it (x_0, y_0).

A component computing Algorithm

- Iterate further, and for all $i = 1, 2, \ldots, M$ evaluate the distance (Euclidean norm in $\mathbb{R}^2$) between $f_\xi^i(x_0, y_0)$ and (x_0, y_0).
- If this distance is less than ε, append the number i to the set J.
- Finally evaluate the greatest common divisor of the elements in J — this is our estimate for the value of k.

The orientation-reversing case

- Let

$$\Phi^{(2)} = \{\xi \in \Phi \mid \delta_L < 0, \delta_R < 0\},$$

  be the subset of Φ for which the BCNF is orientation-reversing.
- The attractor Λ which is again a closure of the unstable manifold of X faces a crisis at $\zeta_0^{(2)} = 0$ where

$$\zeta_0^{(2)} = \phi^-(\xi) = \delta_R - (\delta_R + \tau_R - (1 + \lambda_R^u)\lambda_L^u)\lambda_L^u.$$

The orientation-reversing case

- Now, $\xi \in \Phi^{(2)}$ implies $g(\xi) \in \Phi^{(1)}$, so we again use the preimages of $\phi^{+}(\xi) = 0$ under g to define the region boundaries: Specifically we let

$$\mathcal{R}_0^{(2)} = \left\{ \xi \in \Phi^{(2)} \,\middle|\, \phi^{-}(\xi) > 0, \phi^{+}(g(\xi)) \leq 0, \alpha(\xi) < 0 \right\},$$

$$\mathcal{R}_n^{(2)} = \left\{ \xi \in \Phi^{(2)} \,\middle|\, \phi^{+}\left(g^{n}(\xi)\right) > 0, \phi^{+}\left(g^{n+1}(\xi)\right) \leq 0, \alpha(\xi) < 0 \right\}, \qquad \text{for all } n \geq 1.$$

where

$$\alpha(\xi) = \tau_L \tau_R + (\delta_L - 1)(\delta_R - 1).$$

- This brings us to the proposition

Proposition (Ghosh, McLachlan, & Simpson, 2023, In Prep.)

If $\xi \in \mathcal{R}_n^{(2)}$ with $n \geq 1$, then $g(\xi) \in \mathcal{R}_{n-1}^{(1)}$.

The orientation-reversing case

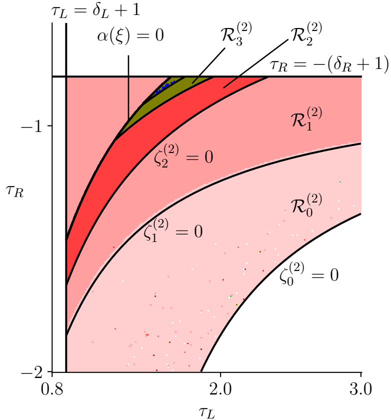
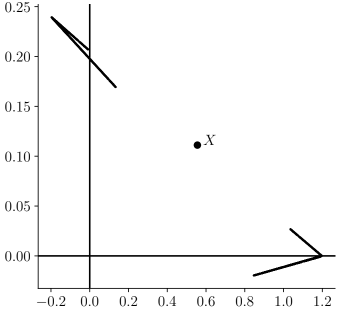
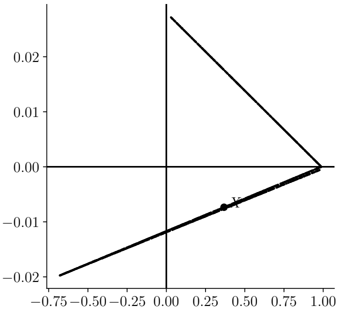


Figure: The sketch of two-dimensional cross-section of $\Phi^{(2)}$, when $\delta_L = -0.1$ and $\delta_R = -0.2$.

The orientation-reversing case



(a) $\xi = \xi_{\mathrm{ex}}^{(2)} \in \mathcal{R}_1^{(2)}$



(b) $\xi = g(\xi_{\mathrm{ex}}^{(2)}) \in \mathcal{R}_0^{(1)}$

The non-invertible case $\delta_L > 0, \delta_R < 0$

- Let

$$\Phi^{(3)} = \{\xi \in \Phi \mid \delta_L > 0, \delta_R < 0\},$$

  meaning the map is invertible.
- In this region an attractor can be destroyed by crossing the homoclinic bifurcation $\phi^+(\xi) = 0$ or the heteroclinic bifurcation $\phi^-(\xi) = 0$.
- we define

$$\phi_{\min}(\xi) = \min[\phi^+(\xi), \phi^-(\xi)].$$

  and

$$\mathcal{R}_n^{(3)} = \left\{ \xi \in \Phi^{(3)} \,\middle|\, \phi_{\min}\left(g^n(\xi)\right) > 0,\ \phi_{\min}\left(g^{n+1}(\xi)\right) \leq 0,\ \alpha(\xi) < 0 \right\},$$

  for all $n \geq 0$.

The non-invertible case $\delta_L > 0, \delta_R < 0$

- This brings us to a new proposition:

Proposition (Ghosh, McLachlan, & Simpson, 2023, In Prep.)

If $\xi \in \mathcal{R}_n^{(3)}$ *with* $n \geq 1$, *then* $g(\xi) \in \mathcal{R}_{n-1}^{(3)}$.

The non-invertible case $\delta_L > 0, \delta_R < 0$

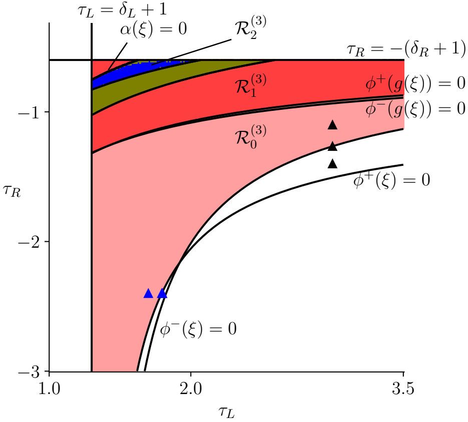
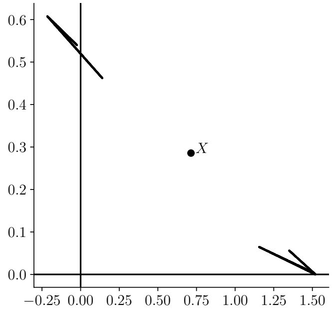
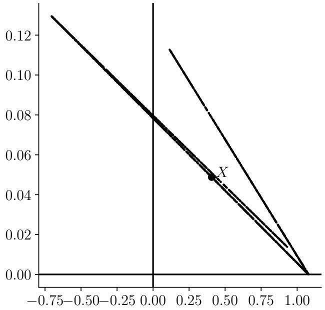


Figure: The sketch of two-dimensional cross-section of $\Phi^{(3)}$, when $\delta_L = 0.3$ and $\delta_R = -0.4$.

The non-invertible case $\delta_L > 0, \delta_R < 0$



(a) $\xi = \xi_{\text{ex}}^{(3)} \in \mathcal{R}_1^{(3)}$



(b) $\xi = g(\xi_{\text{ex}}^{(3)}) \in \mathcal{R}_0^{(3)}$

The non-invertible case $\delta_L < 0, \delta_R > 0$

- It remains for us to consider

$$\Phi^{(4)} = \{\xi \in \Phi \mid \delta_L < 0, \delta_R > 0\},$$

  where the BCNF is again non-invertible.
- In this region the attractor is usually destroyed before the boundaries $\phi^+(\xi) = 0$ and $\phi^-(\xi) = 0$ in a heteroclinic bifurcation that cannot be characterised by an explicit condition on the parameter values.
- Despite the extra complexities in $\Phi^{(4)}$ it still appears that renormalisation is helpful for explaining the bifurcation structure. Let

$$\mathcal{R}_0^{(4)} = \left\{\xi \in \Phi^{(4)} \,\middle|\, \phi_{\min}(\xi) > 0,\ \phi_{\min}(g(\xi)) \le 0,\ \alpha(\xi) < 0\right\}.$$
$$\mathcal{R}_n^{(4)} = \left\{\xi \in \Phi^{(4)} \,\middle|\, \phi_{\min}(g^n(\xi)) > 0,\ \phi_{\min}(g^{n+1}(\xi)) \le 0,\ \alpha(\xi) < 0, \alpha(g(\xi)) < 0\right\}. \tag{1}$$

The non-invertible case $\delta_L < 0, \delta_R > 0$

- This brings us to the new propostion:

Proposition (Ghosh, McLachlan, & Simpson, 2023.)

If $\xi \in \mathcal{R}_n^{(4)}$ with $n \geq 1$, then $g(\xi) \in \mathcal{R}_{n-1}^{(3)}$.

The non-invertible case $\delta_L < 0, \delta_R > 0$

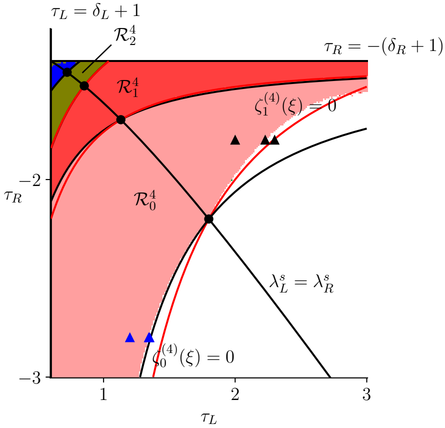


Figure: The sketch of two-dimensional cross-section of $\Phi^{(4)}$, when $\delta_L = -0.4$ and $\delta_R = 0.4$.

The non-invertible case $\delta_L < 0, \delta_R > 0$

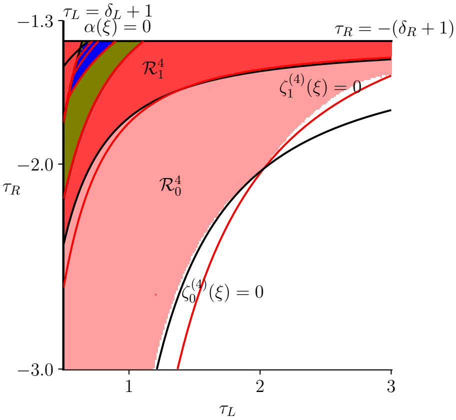
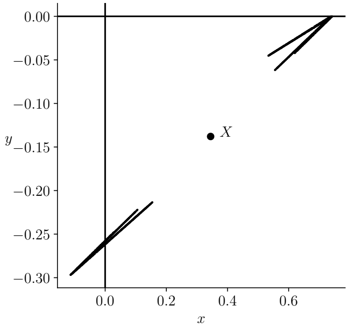
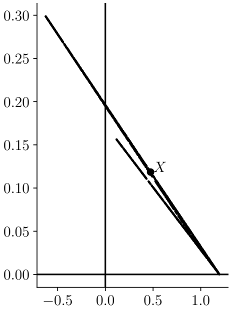


Figure: The sketch of two-dimensional cross-section of $\Phi^{(4)}$, when $\delta_L = -0.5$ and $\delta_R = 0.4$.

The non-invertible case $\delta_L < 0, \delta_R > 0$



(a) $\xi = \xi_{\text{ex}}^{(4)} \in \mathcal{R}_1^{(4)}$



(b) $\xi = g(\xi_{\text{ex}}^{(4)}) \in \mathcal{R}_0^{(3)}$

Summary

- We have used renormalisation to explain how the parameter space Φ_{BYG} is divided into regions according to the number of connected components of an attractor.
- We have further shown how the robust chaos extends more broadly to orientation-reversing and non-invertible piecewise-linear maps.
- We have also extended the application of renormalisation to the orientation-reversing and non-invertible map in a more generalised parameter setting.
- It remains to determine the analogue of the existence of a higher dimensional robust chaos parameter region of the border-collision normal form.

Acknowledgements

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The End

Thank you! Questions?