BIFURCATION STRUCTURE OF ROBUST CHAOS IN A GENERALISED SETTING OF PIECEWISE-LINEAR MAPS.

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Introduction

Here I use geometry to explain robust chaotic dynamics in piecewise-linear (PWL) maps. PWL maps are used for modeling systems with switches, thresholds, and other abrupt events. We study the two-dimensional *border-collision normal form*:

$$f_\xi(x,y) = \begin{cases} \begin{bmatrix} \tau_L & 1 \\ -\delta_L & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}, & x \le 0, \\ \begin{bmatrix} \tau_R & 1 \\ -\delta_R & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}, & x \ge 0, \end{cases}$$

with variables $x, y \in \mathbb{R}$, and parameter vector $\xi = (\tau_L, \delta_L, \tau_R, \delta_R) \in \mathbb{R}^4$.

A renormalisation operator

Although the second iterate f_ξ^2 has four pieces, for many values of ξ only two of these are relevant:

$$f_\xi^2(x,y) = \begin{cases} \begin{bmatrix} \tau_L\tau_R - \delta_L & \tau_R \\ -\delta_R\tau_L & -\delta_R \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} \tau_R + 1 \\ -\delta_R \end{bmatrix}, & x \le 0, \\ \begin{bmatrix} \tau_R^2 - \delta_R & \tau_R \\ -\delta_R\tau_R & -\delta_R \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} \tau_R + 1 \\ -\delta_R \end{bmatrix}, & x \ge 0. \end{cases}$$

Then f_ξ^2 is equivalent to $f_{g(\xi)}$ under a change of coordinates, where g is the *renormalisation operator* defined by

$$\tilde{\tau}_L = \tau_R^2 - 2\delta_R,$$
$$\tilde{\delta}_L = \delta_R^2,$$
$$\tilde{\tau}_R = \tau_L\tau_R - \delta_L - \delta_R,$$
$$\tilde{\delta}_R = \delta_L\delta_R.$$

The classical robust chaos parameter region

Robust chaos refers to the absence of periodic windows. We consider the parameter region

$$\Phi = \left\{ \xi \in \mathbb{R}^4 \,\middle|\, \tau_L > |\delta_L + 1|,\, \tau_R < -|\delta_R + 1| \right\},$$

where f_ξ has two saddle fixed points X and Y, see Fig. 1. Within Φ

$$\Phi_{\rm BYG} = \{ \xi \in \Phi \,|\, \delta_L > 0, \delta_R > 0, \phi_4(\xi) > 0 \},$$

where $\phi_4(\xi) = \delta_R - (\tau_R + \delta_L + \delta_R - (1+\tau_R)\lambda_L^u)\lambda_L^u$, is the classical robust chaos parameter region of [1]. For all $n \ge 0$, define

$$\zeta_n(\xi) = \phi_4(g^n(\xi)),$$

and

$$\mathcal{R}_n = \{ \xi \in \Phi \,|\, \zeta_n(\xi) > 0, \zeta_{n+1}(\xi) \le 0 \},$$

see Fig. 2. Our main result is that for all $n \ge 0$ and $\xi \in \mathcal{R}_n$ the map f_ξ has a chaotic attractor with 2^n connected components [2], e.g. Fig. 3. This was obtained by using the renormalisation operator and reveals previously unknown bifurcation structure within the robust chaos parameter region.

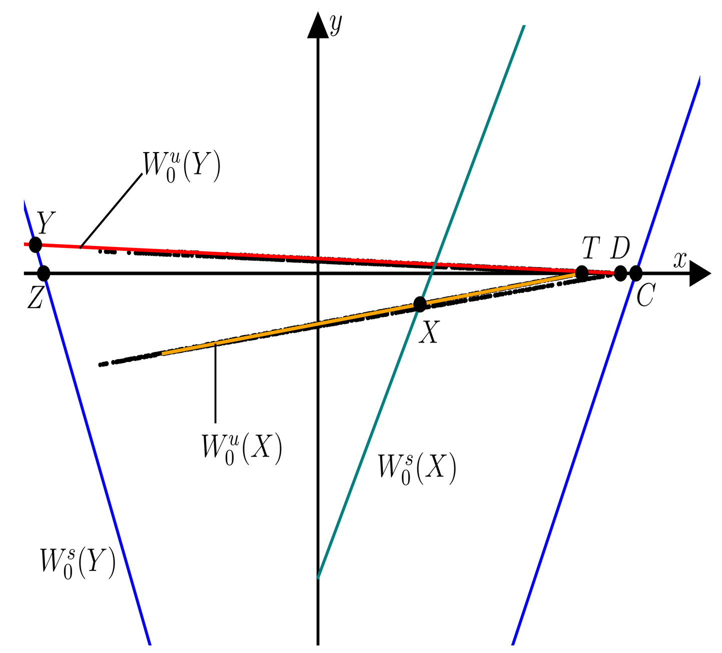

Fig. 1: A phase portrait with $\xi \in \mathcal{R}_0 \subset \Phi_{\rm BYG}$, showing the initial line segments of the stable and unstable manifolds of the fixed points X and Y, as well as an additional segment of the stable manifold of Y. The attractor (computed numerically) is shown in black.

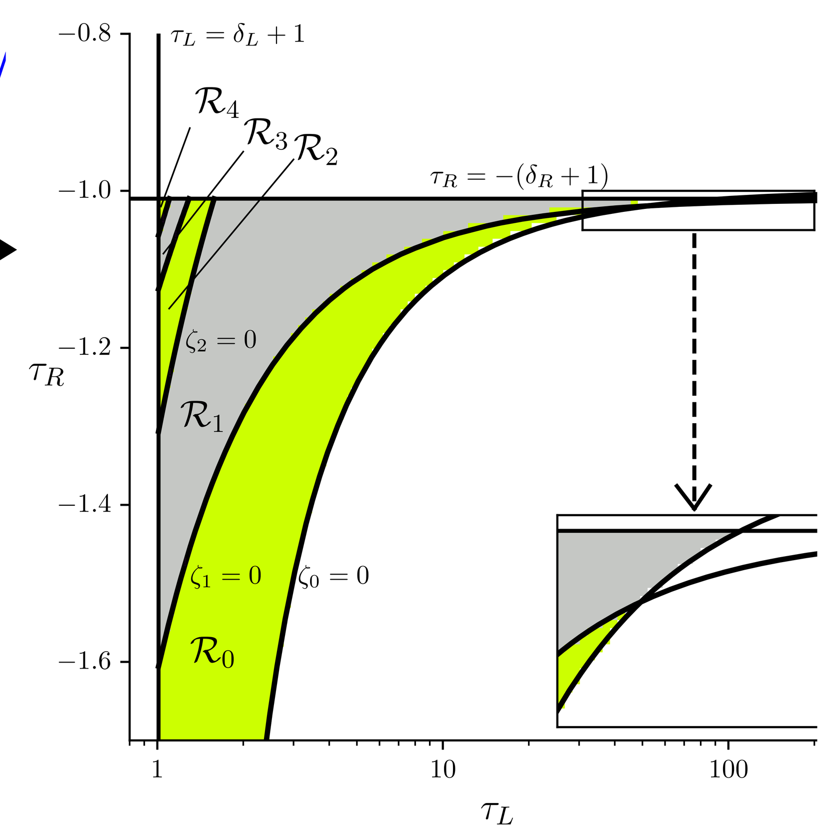

Fig. 2: Two-dimensional cross-sections of the parameter regions $\mathcal{R}_n$, where $\mathcal{R}_n$ is visible for all $n = 0, 1, \ldots, 4$. The region $\Phi_{\rm BYG}$ is bounded by $\tau_L = \delta_L + 1$, $\tau_R = -(\delta_R + 1)$, $\zeta_0 = 0$, $\delta_L > 0$, and $\delta_R > 0$.

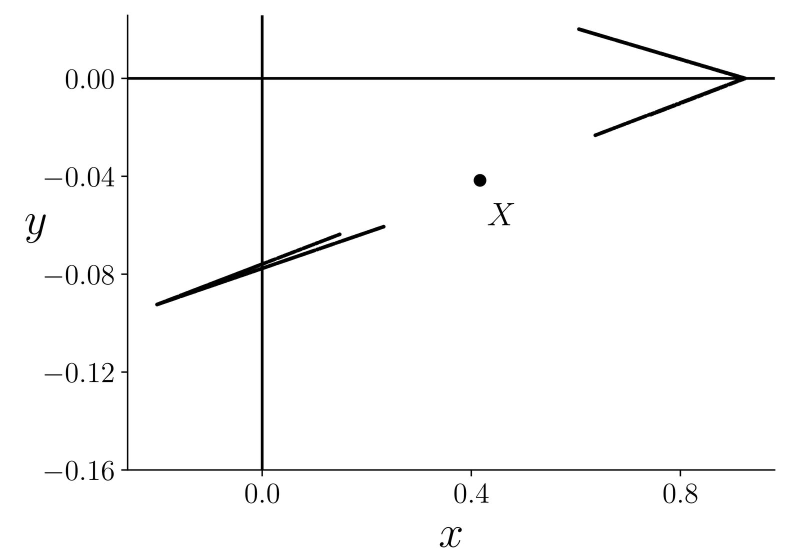

Fig. 3: A phase portrait with $\xi \in \mathcal{R}_1$ where the attractor has two connected components.

Invariant expanding cones

Chaos in $\Phi_{\rm BYG}$ can be proved by constructing an invariant expanding cone, Fig. 4, in tangent space [3]. We have extended this to Φ; Figs. 5 and 6 show parameter values for which we have been able to explicitly construct a trapping region and a cone. For any $\xi \in \Phi_{\rm trap} \cap \Phi_{\rm cone}$, f_ξ has a chaotic attractor [4].

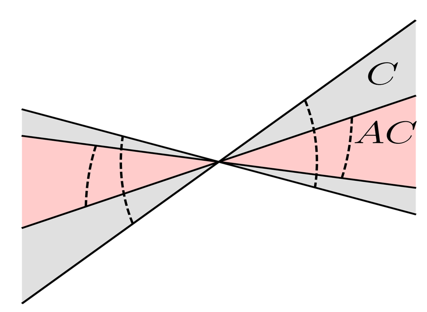

Fig. 4: A sketch of an invariant expanding cone C and its image $AC = \{Av | v \in C\}$, given $A \in \mathbb{R}^{2\times2}$.

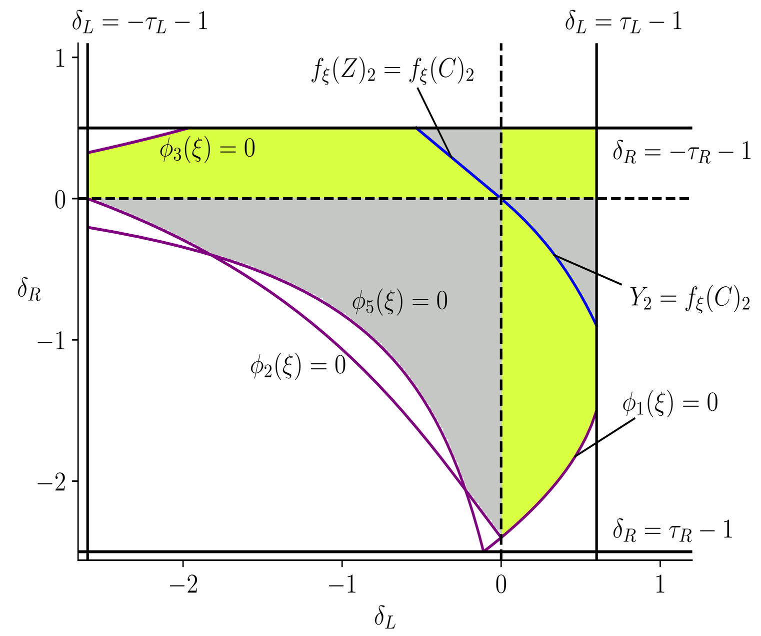

Fig. 5: A 2D cross-section of $\Phi_{\rm trap}$, showing curves $\phi_i(\xi) = 0$, for $\tau_L = 1.6$ and $\tau_R = -1.5$.

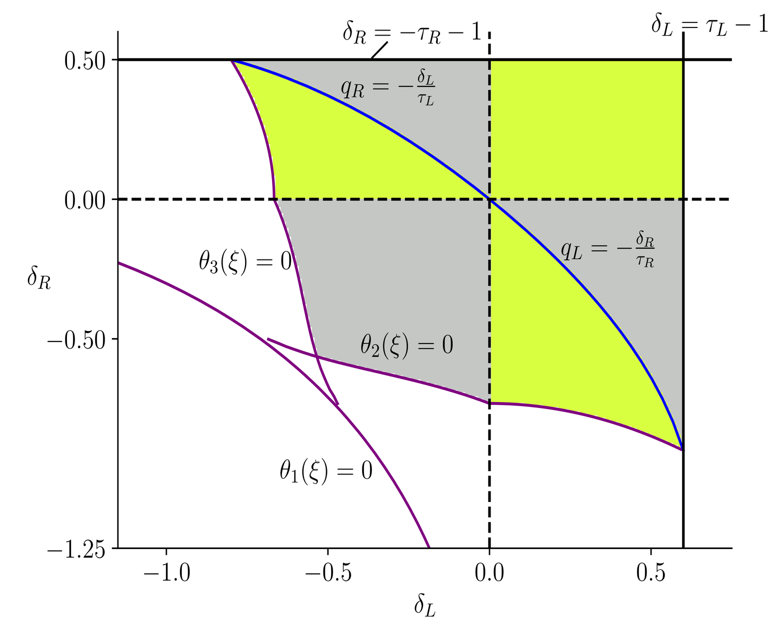

Fig. 6: A 2D cross-section of $\Phi_{\rm cone}$, showing curves $\theta_i(\xi) = 0$, for $\tau_L = 1.6$ and $\tau_R = -1.5$.

Extension to higher dimensions

In the N-dimensional setting, suppose the fixed point Y has exactly one unstable eigenvalue $\lambda_L^1 > 1$ and the fixed point X has exactly one unstable eigenvalue $\lambda_R^1 < -1$. We have been able to construct an N-dimensional trapping region in an open region of parameter space, see Fig. 7.

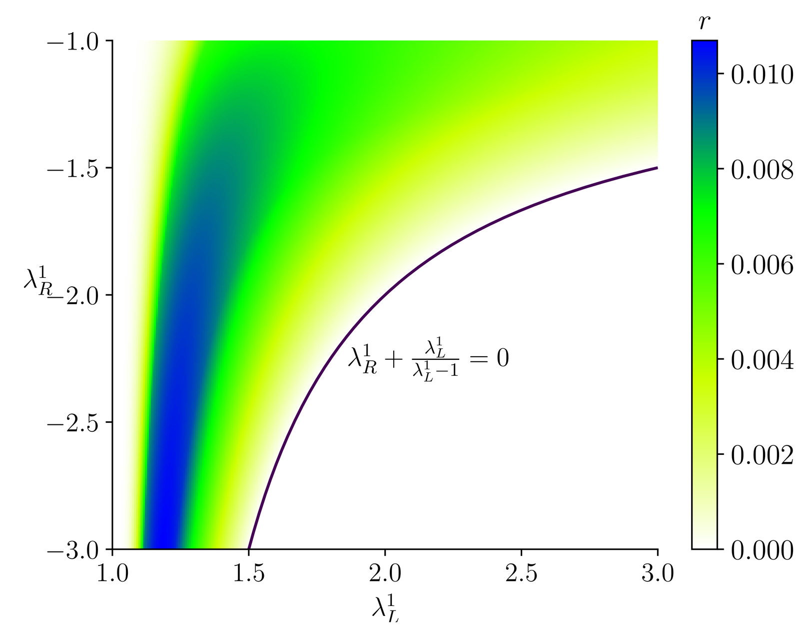

Fig. 7: Our trapping region construction is valid when the absolute values of the stable eigenvalues are all less than the indicated value of r.

Acknowledgements

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References

[1] S. Banerjee, J.A. Yorke, and C. Grebogi. *Phys. Rev. Lett.*, 80(14):3049–3052, 1998.

[2] I. Ghosh, and D.J.W. Simpson. *Int. J. Bifurcation Chaos*, 32(12):2250181, 2022.

[3] P.A. Glendinning, and D.J.W. Simpson. *Discrete Contin. Dyn. Syst.*, 41(7):3367–3387, 2021.

[4] I. Ghosh, R. McLachlan, and D.J.W. Simpson. A generalised construction for robust chaos in two-dimensional piecewise-linear maps. *In preparation.*