Renormalisation of the Two-Dimensional Border-Collision Normal Form

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Border-collision normal form

- Piecewise-linear maps arise when modeling systems with switches, thresholds and other abrupt events.
- In our project, we study the two-dimensional *border-collision normal form* [H.E. Nusse and J.A. Yorke, 1992], given by

$$f_\xi(x,y) = \begin{cases} \begin{bmatrix} \tau_L & 1 \\ -\delta_L & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}, & x \le 0, \\ \begin{bmatrix} \tau_R & 1 \\ -\delta_R & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}, & x \ge 0. \end{cases}$$

- Here $(x,y) \in \mathbb{R}^2$, and $\xi = (\tau_L, \delta_L, \tau_R, \delta_R) \in \mathbb{R}^4$ are the parameters.

Banerjee-Yorke-Grebogi region in parameter space

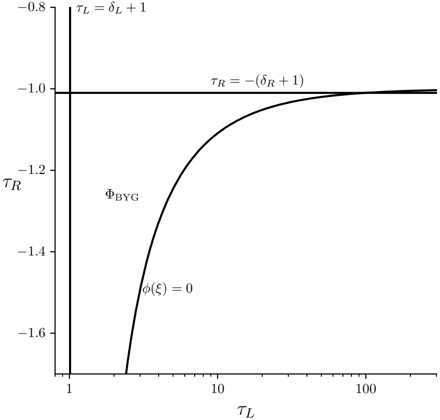


Figure: Sketch of the parameter region Φ_{BYG} [S. Banerjee, J.A. Yorke, and C. Grebogi. Robust chaos. *Phys. Rev. Lett.*, 80(14):3049– 3052, 1998.], with $\delta_L = \delta_R = 0.01$.

Phase portrait of a chaotic attractor

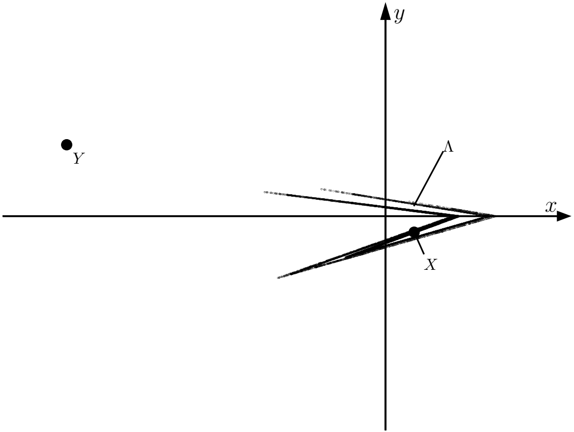


Figure: A sketch of the phase portrait of f_ξ with $\xi \in \Phi_{\mathrm{BYG}}$.

Renormalisation I

- Renormalisation involves showing that, for some member of a family of maps, a higher iterate or induced map is conjugate to different member of this family of maps.
- The *renormalisation* technique (Feigenbaum, 1970's) proves that the bifurcation values in period-doubling cascades for one-dimensional unimodal maps converge at a constant rate ($F \simeq 4.669\ldots$), which is universal. For example, the *logistic map* given by

$$x_{n+1} = \mu x_n(1 - x_n),$$

  has the following bifurcation diagram.

Renormalisation II

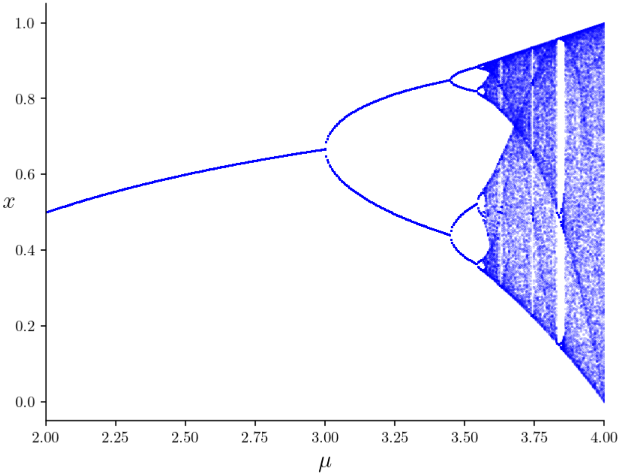


Figure: Bifurcation diagram for the logistic map.

Renormalisation IV

- Let $\mathfrak{U}$ denote the collection of all unimodal maps $f : [-1, 1] \to [-1, 1]$, with maximum at $x = 0$, and with $f(0) = 1$. Then, the *renormalisation operator* $\mathfrak{R} : \mathfrak{U} \to \mathfrak{U}$ is given by,
$$(\mathfrak{R}f)(x) = -\frac{1}{a} f^2(-ax),$$
provided, $a = -f(1)$, $b = f(a)$, $0 < a < b < 1$ and $f(b) < a$.
- The fixed point of $\mathfrak{R}$ is *hyperbolic*. One of its eigenvalues has modulus greater than 1, and this eigenvalue is Feigenbaum's constant F [Feigenbaum, 1975].

Renormalisation operator I

- Although the second iterate f_ξ^2 has four pieces, relevant dynamics arise in only two of these. We have

$$f_\xi^2(x,y) = \begin{cases} \begin{bmatrix} \tau_L\tau_R - \delta_L & \tau_R \\ -\delta_R\tau_L & -\delta_R \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} \tau_R + 1 \\ -\delta_R \end{bmatrix}, & x \le 0, \\ \begin{bmatrix} \tau_R^2 - \delta_R & \tau_R \\ -\delta_R\tau_R & -\delta_R \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} \tau_R + 1 \\ -\delta_R \end{bmatrix}, & x \ge 0. \end{cases}$$

- Now f_ξ^2 can be transformed to $f_{g(\xi)}$, where g is the *renormalisation operator* $g : \mathbb{R}^4 \to \mathbb{R}^4$, given by

$$\begin{aligned} \tilde{\tau}_L &= \tau_R^2 - 2\delta_R, \\ \tilde{\delta}_L &= \delta_R^2, \\ \tilde{\tau}_R &= \tau_L\tau_R - \delta_L - \delta_R, \\ \tilde{\delta}_R &= \delta_L\delta_R. \end{aligned}$$

Renormalisation operator II

- We perform a coordinate change to put f_ξ^2 in the normal form :

$$\begin{bmatrix} \tilde{x}' \\ \tilde{y}' \end{bmatrix} = \begin{cases} \begin{bmatrix} \tilde{\tau}_L & 1 \\ -\tilde{\delta}_L & 0 \end{bmatrix} \begin{bmatrix} \tilde{x} \\ \tilde{y} \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}, & \tilde{x} \leq 0, \\ \begin{bmatrix} \tilde{\tau}_R & 1 \\ -\tilde{\delta}_R & 0 \end{bmatrix} \begin{bmatrix} \tilde{x} \\ \tilde{y} \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}, & \tilde{x} \geq 0. \end{cases}$$

Results I

- We consider the parameter region

$$\Phi = \left\{ \xi \in \mathbb{R}^4 \middle| \tau_L > \delta_L + 1, \delta_L > 0, \tau_R < -(\delta_R + 1), \delta_R > 0 \right\}.$$

- The stable and the unstable manifolds of the fixed point Y intersect if and only if $\phi(\xi) \leq 0$, where,

$$\phi(\xi) = \delta_R - (\tau_R + \delta_L + \delta_R - (1 + \tau_R)\lambda_L^u)\lambda_L^u.$$

- Banerjee, Yorke and Grebogi observed that an attractor is often destroyed at $\phi(\xi) = 0$ which is a homoclinic bifurcation, and thus focused their attention on the region

$$\Phi_{\mathrm{BYG}} = \{\xi \in \Phi | \phi(\xi) > 0\}.$$

Results II

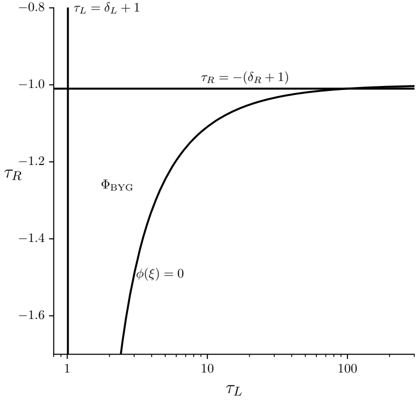


Figure: Sketch of the parameter region Φ_{BYG}, with $\delta_L = \delta_R = 0.01$.

Results III

- For all $n \geq 0$ let
$$\zeta_n(\xi) = \phi(g^n(\xi)),$$
where $\zeta_n(\xi) = 0$ is the n^{th} preimage of $\phi(\xi) = 0$ under the operator g.
- The regions $\mathcal{R}_n$ for all $n \geq 0$ are thus generated, having the form:
$$\mathcal{R}_n = \{\xi \in \Phi | \zeta_n(\xi) > 0, \zeta_{n+1}(\xi) \leq 0\}.$$

Results IV

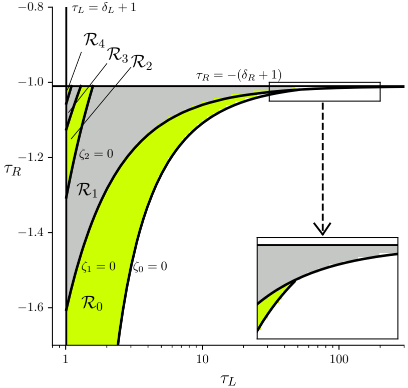


Figure: The sketch of two dimensional cross-section of $\mathcal{R}_n$ when $\delta_L = \delta_R = 0.01$.

Results V

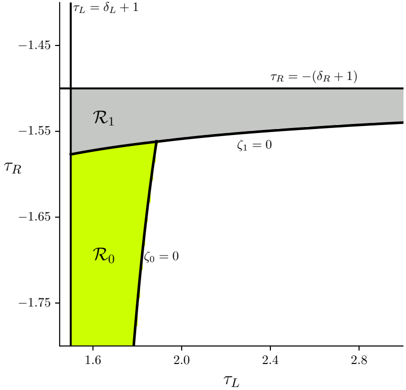


Figure: The sketch of two dimensional cross-section of $\mathcal{R}_n$ when $\delta_L = \delta_R = 0.5$.

Results VI

Theorem

The $\mathcal{R}_n$ are non-empty, mutually disjoint, and converge to the fixed point $(1, 0, -1, 0)$ *as* $n \to \infty$. *Moreover,*

$$\Phi_{\mathrm{BYG}} \subset \bigcup_{n=0}^{\infty} \mathcal{R}_n.$$

- Let,

$$\Lambda(\xi) = \mathrm{cl}(W^u(X)).$$

Theorem

For the map f_ξ *with any* $\xi \in \mathcal{R}_0$, $\Lambda(\xi)$ *is bounded, connected, and invariant. Moreover,* $\Lambda(\xi)$ *is chaotic (positive Lyapunov exponent).*

Results VII

Theorem

For any $\xi \in \mathcal{R}_n$ where $n \geq 0$, $g^n(\xi) \in \mathcal{R}_0$ and there exist mutually disjoint sets $S_0, S_1, \ldots, S_{2^n-1} \subset \mathbb{R}^2$ such that $f_\xi(S_i) = S_{(i+1) \text{ mod } 2^n}$ and

$$f_\xi^{2^n}|_{S_i} \text{ is affinely conjugate to } f_{g^n(\xi)}|_{\Lambda(g^n(\xi))}$$

for each $i \in \{0, 1, \ldots, 2^n - 1\}$. Moreover,

$$\bigcup_{i=0}^{2^n-1} S_i = \mathrm{cl}(W^u(\gamma_n)),$$

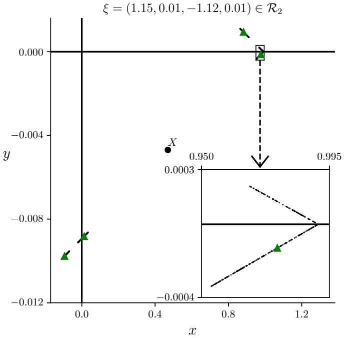
where γ_n is a saddle-type periodic solution of our map f_ξ having the symbolic itinerary $\mathcal{F}^n(R)$ given by Table 1.

Results VIII

n	$\mathcal{F}^n(\mathcal{W})$
0	R
1	LR
2	$RRLR$
3	$LRLRRRLR$
4	$RRLRRRLRLRLRRRLR$

Table: The first 5 words in the sequence generated by repeatedly applying the substitution rule $(L, R) \mapsto (RR, LR)$ to $\mathcal{W} = R$.

Results IX



$\xi = (1.15, 0.01, -1.12, 0.01) \in \mathcal{R}_2$
0.000
−0.004
−0.008
−0.012
y
0.0
0.4
0.8
1.2
x
X
0.950
0.995
0.0003
−0.0004
$\xi = g^2(1.15, 0.01, -1.12, 0.01) \in \mathcal{R}_0$
0
-10^{-8}
y
−0.6
−0.2
0.2
0.6
1.0
x
X

Summary

- We have used renormalization to explain how the parameter space Φ_{BYG} is divided into regions according to the number of connected components of an attractor.
- It remains to better understand the attractor $\mathcal{R}_0$ more and determine the analogue of Φ_{BYG} for higher dimensional maps.
- Our results have been submitted to *Int. J. Bifurcation Chaos* (arXiv:2109.09242, 2021).

Acknowledgements

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The End

Thank you! Questions?