Bifurcation Structure within Robust Chaos for Piecewise-Linear Maps.

I. Ghosh, R. McLachlan, D.J.W. Simpson

School of Mathematical and Computational Sciences, Massey University, Palmerston North, New Zealand

Introduction

Here I use geometry to explain robust chaotic dynamics in piecewise-linear (PWL) maps. PWL maps are used for modeling systems with switches, thresholds, and other abrupt events. We study the two-dimensional *border-collision normal form*:

$$f_\xi(x,y) = \begin{cases} \begin{bmatrix} \tau_L & 1 \\ -\delta_L & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}, & x \le 0, \\ \begin{bmatrix} \tau_R & 1 \\ -\delta_R & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}, & x \ge 0, \end{cases}$$

with variables $x, y \in \mathbb{R}$, and parameter vector $\xi = (\tau_L, \delta_L, \tau_R, \delta_R) \in \mathbb{R}^4$.

A renormalisation operator

Although the second iterate f_ξ^2 has four pieces, for many values of ξ only two of these are relevant:

$$f_\xi^2(x,y) = \begin{cases} \begin{bmatrix} \tau_L\tau_R - \delta_L & \tau_R \\ -\delta_R\tau_L & -\delta_R \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} \tau_R + 1 \\ -\delta_R \end{bmatrix}, & x \le 0, \\ \begin{bmatrix} \tau_R^2 - \delta_R & \tau_R \\ -\delta_R\tau_R & -\delta_R \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} \tau_R + 1 \\ -\delta_R \end{bmatrix}, & x \ge 0. \end{cases}$$

Then f_ξ^2 is equivalent to $f_{g(\xi)}$ under a change of coordinates, where g is the *renormalisation operator* defined by

$$\begin{aligned} \tilde{\tau}_L &= \tau_R^2 - 2\delta_R, \\ \tilde{\delta}_L &= \delta_R^2, \\ \tilde{\tau}_R &= \tau_L\tau_R - \delta_L - \delta_R, \\ \tilde{\delta}_R &= \delta_L\delta_R. \end{aligned}$$

The classical robust chaos parameter region

Robust chaos refers to the absence of periodic windows. We consider the parameter region

$$\Phi = \left\{ \xi \in \mathbb{R}^4 \,\middle|\, \tau_L > |\delta_L + 1|,\ \tau_R < |\delta_R + 1| \right\},$$

where f_ξ has two saddle fixed points X and Y, see Fig. 1. Within Φ

$$\Phi_{\mathrm{BYG}} = \left\{ \xi \in \Phi \,\middle|\, \delta_L > 0, \delta_R > 0, \phi_4(\xi) > 0 \right\},$$

where $\phi_4(\xi) = \delta_R - (\tau_R + \delta_L + \delta_R - (1 + \tau_R)\lambda_L^u)\lambda_L^u$, is the classical robust chaos parameter region of [1]. For all $n \ge 0$, define

$$\zeta_n(\xi) = \phi_4(g^n(\xi)),$$

and

$$\mathcal{R}_n = \{\xi \in \Phi \,|\, \zeta_n(\xi) > 0, \zeta_{n+1}(\xi) \le 0\},$$

see Fig. 2. Our main result is that for all $n \ge 0$ and $\xi \in \mathcal{R}_n$ the map f_ξ has a chaotic attractor with 2^n connected components [2], e.g. Fig. 3. This was obtained by using the renormalisation operator and reveals previously unknown bifurcation structure within the robust chaos parameter region.

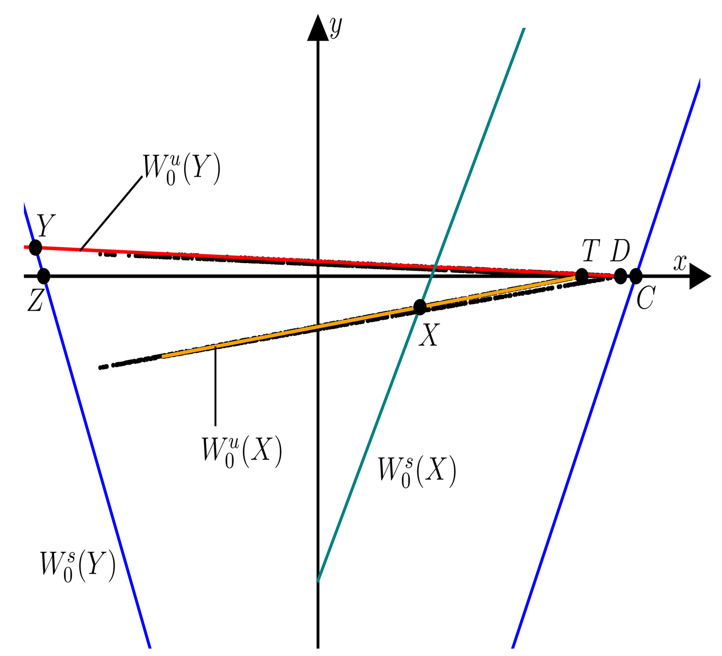

Fig. 1: A phase portrait with $\xi \in \mathcal{R}_0 \subset \Phi_{\mathrm{BYG}}$, showing the initial line segments of the stable and unstable manifolds of the fixed points X and Y, as well as an additional segment of the stable manifold of Y. The attractor (computed numerically) is shown in black.

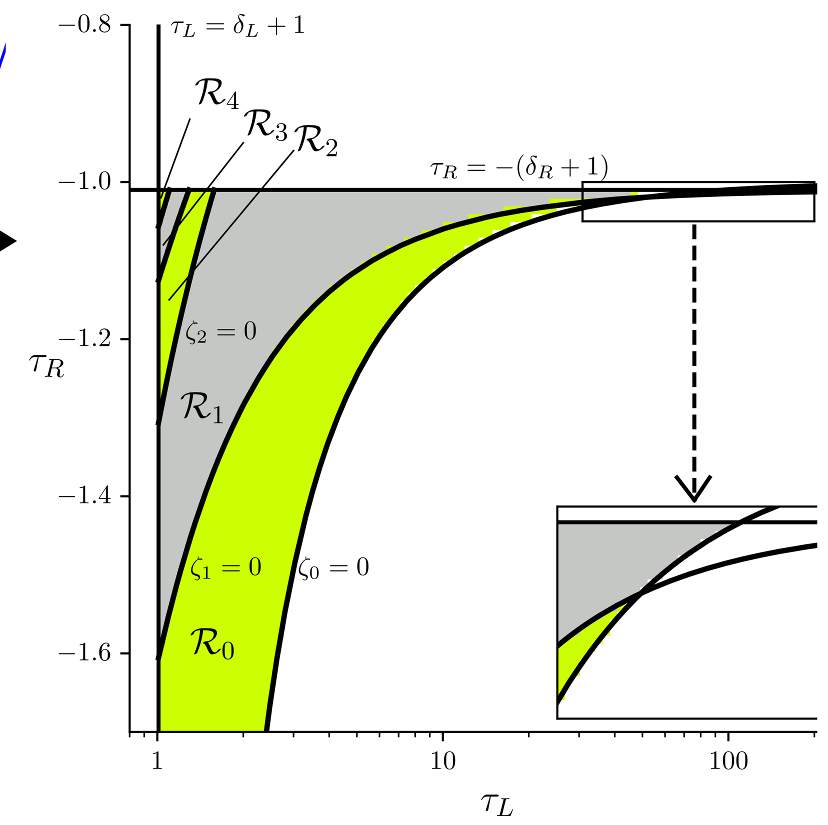

Fig. 2: Two-dimensional cross-sections of the parameter regions $\mathcal{R}_n$, where $\mathcal{R}_n$ is visible for all $n = 0, 1, \ldots, 4$. The region Φ_{BYG} is bounded by $\tau_L = \delta_L + 1$, $\tau_R = -(\delta_R + 1)$, $\zeta_0 = 0$, $\delta_L > 0$, and $\delta_R > 0$.

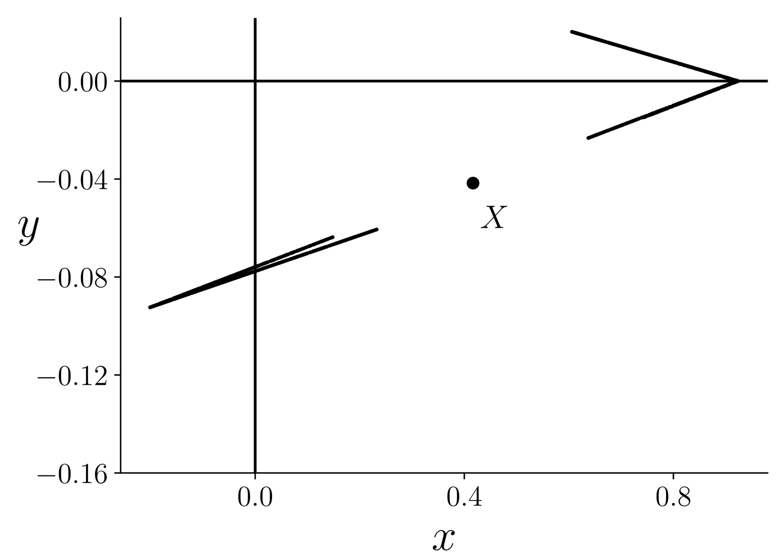

Fig. 3: A phase portrait with $\xi \in \mathcal{R}_1$ where the attractor has two connected components.

Invariant expanding cones

Chaos in Φ_{BYG} can be proved by constructing an invariant expanding cone, Fig. 4, in tangent space [3]. We have extended this to Φ; Figs. 5 and 6 show parameter values for which we have been able to explicitly construct a trapping region and a cone. For any $\xi \in \Phi_{\mathrm{trap}} \cap \Phi_{\mathrm{cone}}$, f_ξ has a chaotic attractor [4].

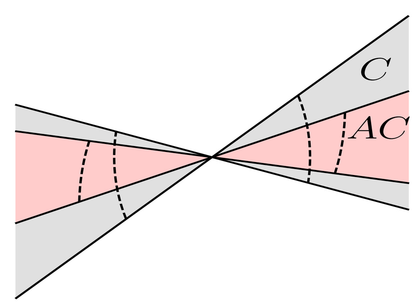

Fig. 4: A sketch of an invariant expanding cone C and its image $AC = \{Av | v \in C\}$, given $A \in \mathbb{R}^{2\times 2}$.

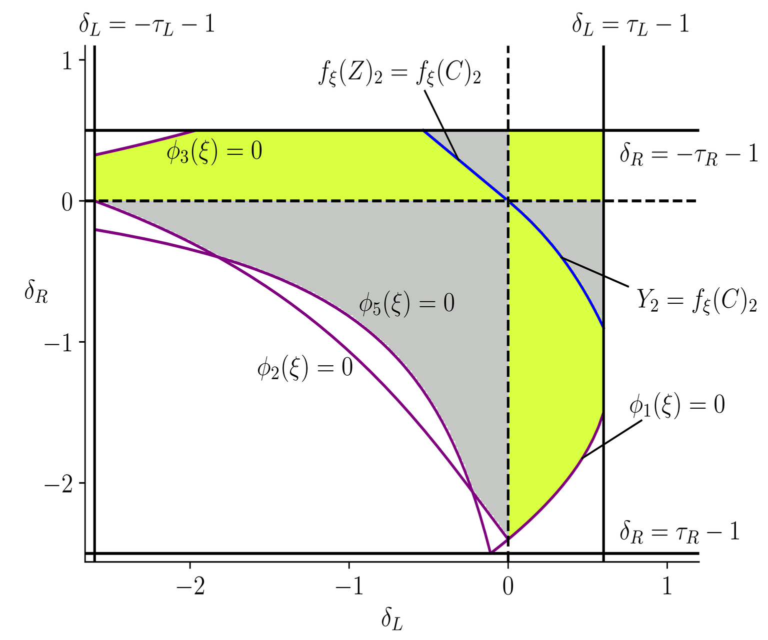

Fig. 5: A 2D cross-section of Φ_{trap}, showing curves $\phi_i(\xi) = 0$, for $\tau_L = 1.6$ and $\tau_R = -1.5$.

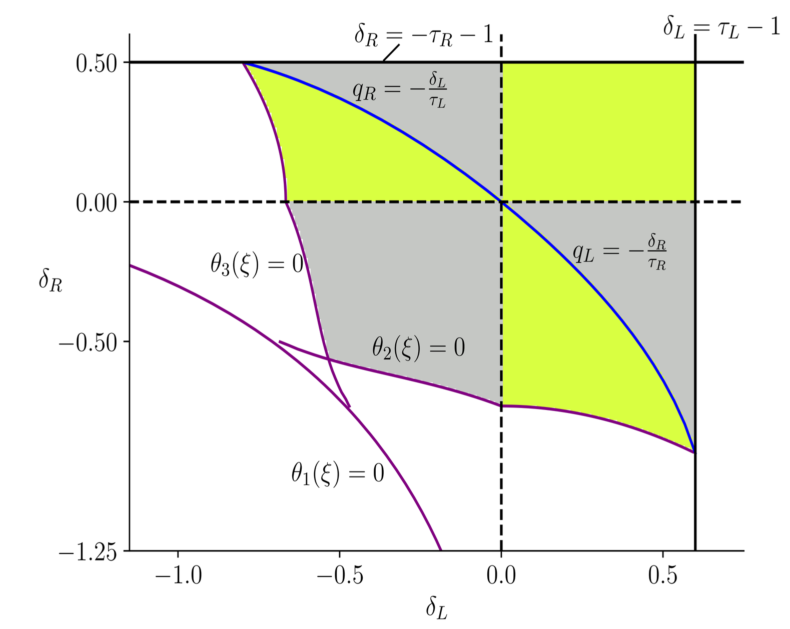

Fig. 6: A 2D cross-section of Φ_{cone}, showing curves $\theta_i(\xi) = 0$, for $\tau_L = 1.6$ and $\tau_R = -1.5$.

Extension to higher dimensions

In the N-dimensional setting, suppose the fixed point Y has exactly one unstable eigenvalue $\lambda_L^1 > 1$ and the fixed point X has exactly one unstable eigenvalue $\lambda_R^1 < -1$. We have been able to construct an N-dimensional trapping region in an open region of parameter space, see Fig. 7.

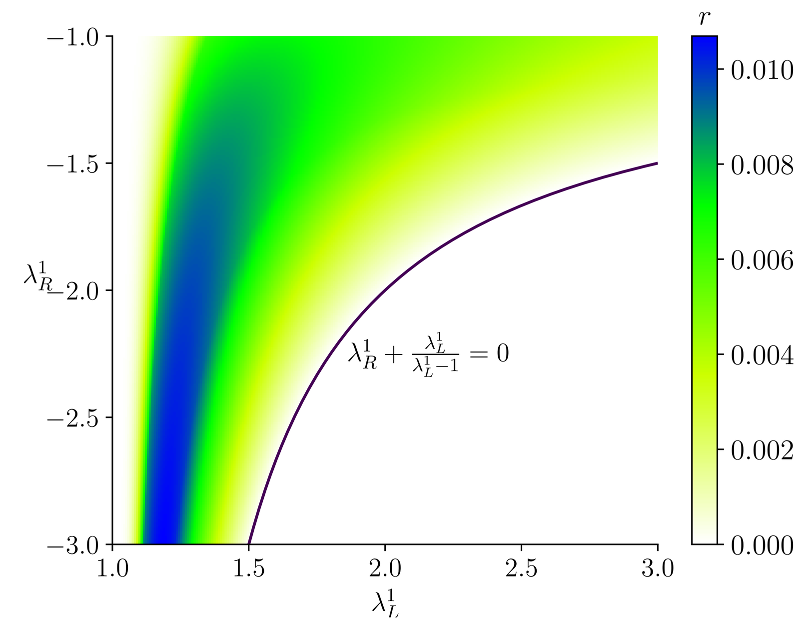

Fig. 7: Our trapping region construction is valid when the absolute values of the stable eigenvalues are all less than the indicated value of r.

Acknowledgements

The authors were supported by Marsden Fund contract MAU1809, managed by Royal Society Te Apārangi.

References

[1] S. Banerjee, J.A. Yorke, and C. Grebogi. *Phys. Rev. Lett.*, 80(14):3049–3052, 1998.

[2] I. Ghosh, and D.J.W. Simpson. *Int. J. Bifurcation Chaos*, 32(12):2250181, 2022.

[3] P.A. Glendinning, and D.J.W. Simpson. *Discrete Contin. Dyn. Syst.*, 41(7):3367–3387, 2021.

[4] I. Ghosh, R. McLachlan, and D.J.W. Simpson. A generalised construction for robust chaos in two-dimensional piecewise-linear maps. *In preparation*.