Time series analysis for coupled neurons

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’Bout me!

1. B.Sc. and M.Sc. in Physics (2015–2020)

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4. Current postdoc in Mathematical Neuroscience (2025—Present)

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An Coláiste Ollscoile, Baile Átha Cliath

Dynamical systems (ODEs)

1. ODEs : Ordinary Differential Equations.

2. Rate of change of a physical quantity over time.

3. Generates a data of time series, given an initial time stamp.

$$\dot{\mathbf{x}} = \frac{d\mathbf{x}}{dt} = f(\mathbf{x}, t)$$

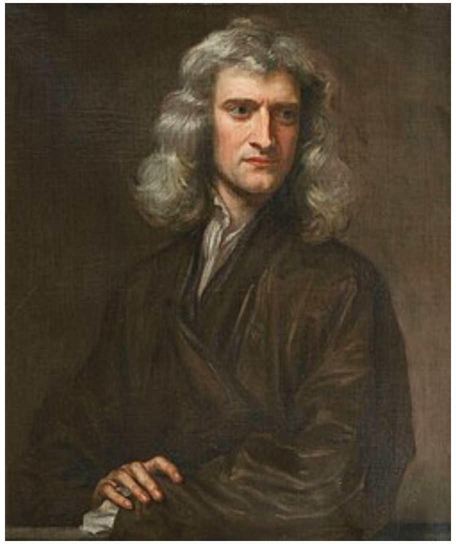



Fig. Newton and Leibniz (Wikipedia)

Neurons

1. Neurons are the fundamental units of the nervous system.

2. Billions of neurons couple through ‘synapses’ to form a cluster of a highly complex neural mass.

3. Their mechanism evolves in time.

4. Thus can be perceived as a `dynamical system’.

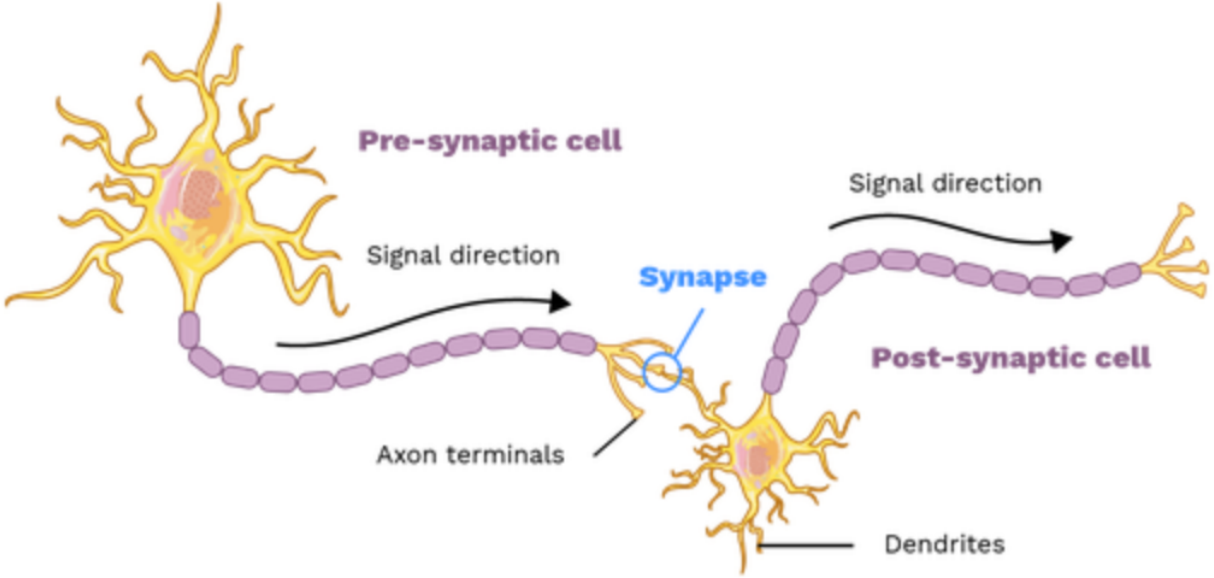


Fig. A typical synapse (theory.labster.com/synapses/)

Chaos

1. In popular term a ‘state of disorder’.

2. In mathematical term, it must be sensitive to initial conditions and have a dense orbit in the phase space.

3. Chaotic systems behave predictably in the beginning before becoming random.

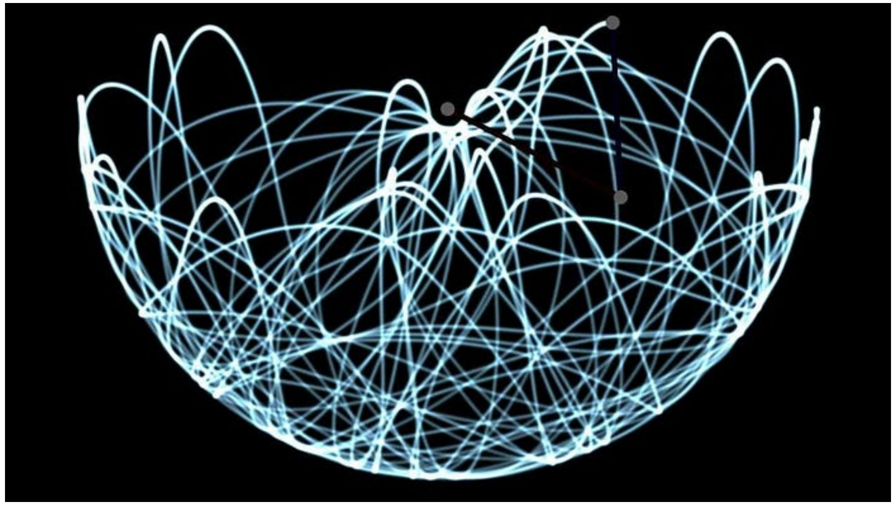


Fig. Double-rod pendulum exhibiting chaos (Taken from https://medium.com/@bharatambati/how-the-double-pendulum-creates-simple-chaos-ac49a297fb4d)

Packages

Single neuron (Schaeffer & Cain, 2018)

1. Simple mathematical model.
2. Parameters selected from empirical experiments.
3. Captures realistic bursting in neurons.
4. Portrays a battery of complex dynamics.
5. Use `solve_ivp()`function from `scipy.integrate` suite to solve initial value problem.

$$\dot{x} = f(x, y, I) = x^2(1 - x) - y + I,$$

$$\dot{y} = g(x, y, I) = Ae^{\alpha x} - \gamma y,$$

$$\dot{I} = h(x, y, I) = \varepsilon \left[\frac{1}{60} \left\{ 1 + \tanh\left(\frac{0.05 - x}{0.001} \right) \right\} - I \right]$$

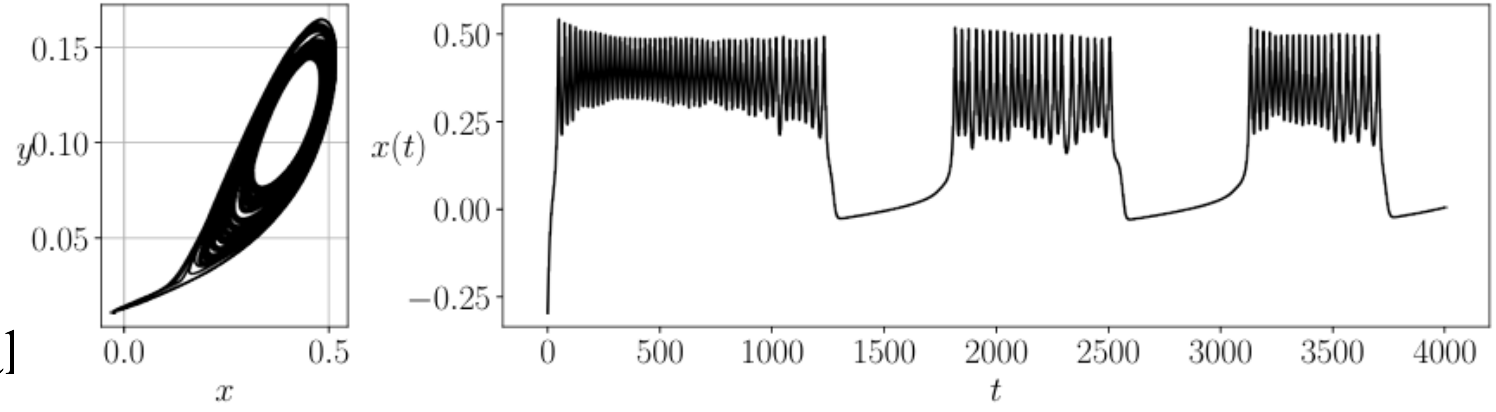


Fig. Phase portrait and time series

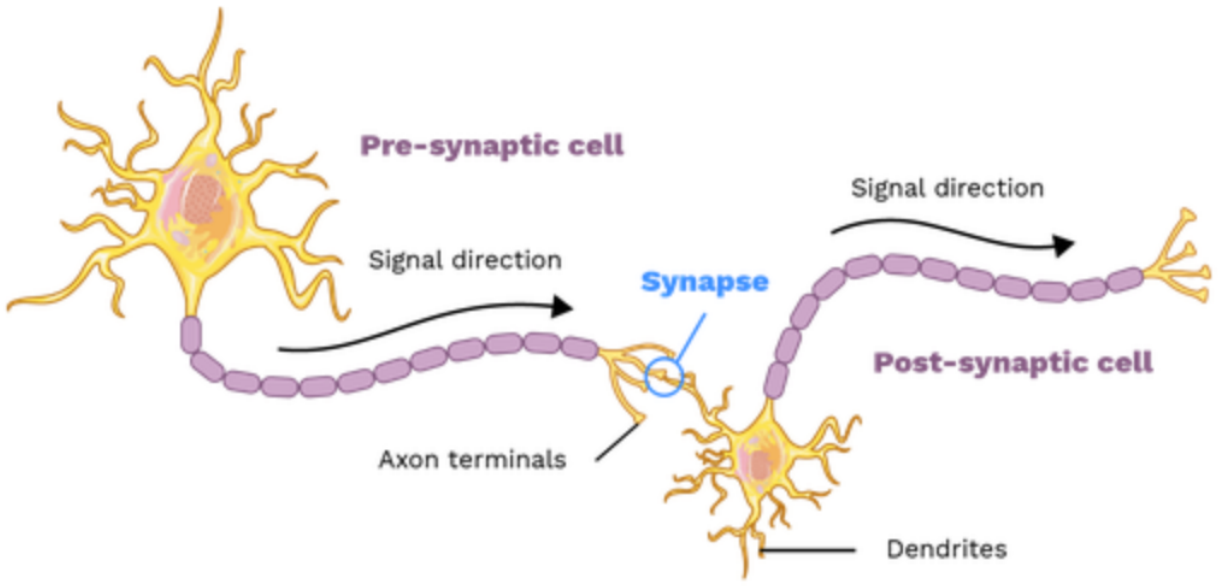
Simulate a single neuron (code snippet)

```
## Parameters
A = 0.0041
alpha=5.276
gamma = 0.315
epsilon = 0.0005

## Define the function of differential equations
def system(t, vars):
    x1, y1, I1= vars
    dx1dt = x1**2 * (1 - x1) - y1 + I1
    dy1dt = A * np.exp(alpha * x1) - gamma * y1
    dI1dt = epsilon*(1/60*(1+np.tanh((0.05-x1)/0.001)) - I1)

    return [dx1dt, dy1dt, dI1dt]

## Initial conditions
x1_0 = np.random.uniform(low=-1, high=1)
y1_0 = 0.1
I1_0 = 0.019

initial_conditions = [x1_0, y1_0, I1_0]
```

```
## Time span for the solution
t_span = (0, 4000)
t_eval = np.linspace(t_span[0], t_span[1], 50000)

## Solve
solution = solve_ivp(system, t_span, initial_conditions, t_eval=t_eval, method='RK45')

## Extract solutions
x1_sol = solution.y[0]
y1_sol = solution.y[1]
I1_sol = solution.y[2]


tt = solution.t
print("done")
```

For the time integration of the differential equations, we use method = ‘RK45’ which is the explicit Runge-Kutta scheme of order 5(4).

Coupled neurons



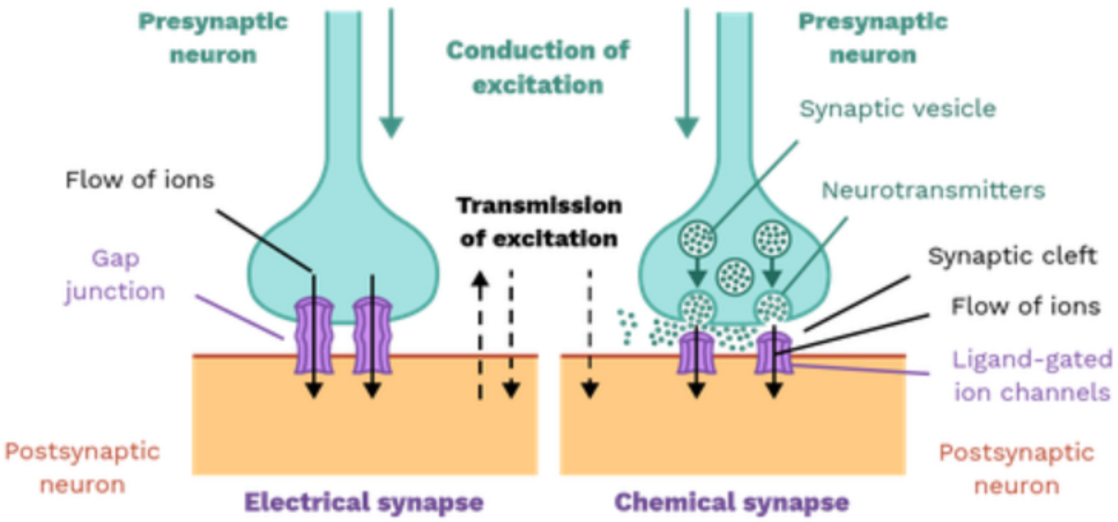


Fig. Coupled neurons (https://theory.labster.com/synapses/), and typical electrical and chemical synapses (theory.labster.com/electrical-synapses/)

Toy models of coupled neurons

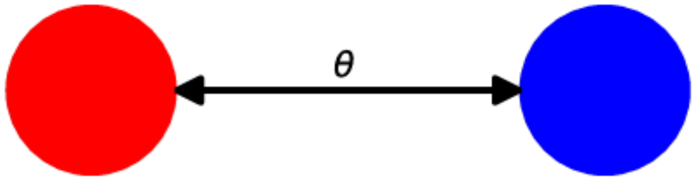


Fig. Electrical (gap-junction) coupling

$$\dot{x}_i = f(x_i, y_i, I_i) + \sum_{j \in B(i)} \theta(x_j - x_i),$$
$$\dot{y}_i = g(x_i, y_i, I_i),$$
$$\dot{I}_i = h(x_i, y_i, I_i),$$

Coupling term

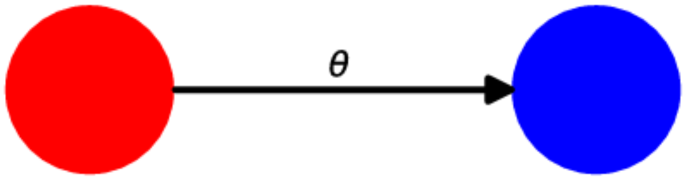


Fig. Chemical coupling

$$\dot{x}_1 = f(x_1, y_1, I_1),$$
$$\dot{x}_2 = f(x_2, y_2, I_2) + \theta \frac{v_s - x_2}{1 + \exp\{-\lambda(x_1 - q)\}},$$
$$\dot{y}_i = g(x_i, y_i, I_i),$$
$$\dot{I}_i = h(x_i, y_i, I_i),$$

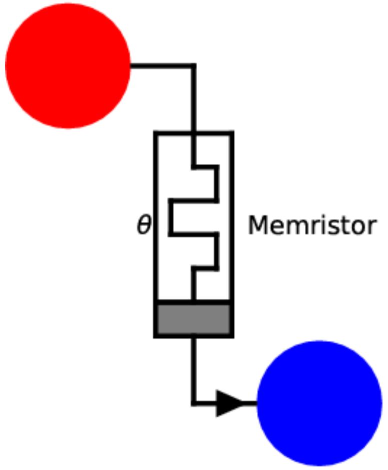


Fig. Electromagnetic coupling

$$\dot{x}_1 = f(x_1, y_1, I_1) + \theta\rho(\phi)(x_2 - x_1),$$
$$\dot{x}_2 = f(x_2, y_2, I_2) + \theta\rho(\phi)(x_1 - x_2),$$
$$\dot{y}_i = g(x_i, y_i, I_i),$$
$$\dot{I}_i = h(x_i, y_i, I_i), \qquad i = 1, 2,$$
$$\dot{\phi} = \theta(x_1 - x_2)$$

Simulating coupled neurons

```python
def system(t, vars):
    x1, y1, I1, x2, y2, I2= vars
    dx1dt = x1**2 * (1 - x1) - y1 + I1+ theta*(x2-x1)
    dy1dt = A * np.exp(alpha * x1) - gamma * y1
    dI1dt = epsilon*(1/60*(1+np.tanh((0.05-x1)/0.001)) - I1)
    dx2dt = x2**2 * (1 - x2) - y2 + I2 + theta*(x1-x2)
    dy2dt = A * np.exp(alpha * x2) - gamma * y2
    dI2dt = epsilon*(1/60*(1+np.tanh((0.05-x2)/0.001)) - I2)

    return [dx1dt, dy1dt, dI1dt, dx2dt, dy2dt, dI2dt]
```

```python
def system(t, vars):
    x1, y1, I1, x2, y2, I2= vars
    dx1dt = x1**2 * (1 - x1) - y1 + I1
    dy1dt = A * np.exp(alpha * x1) - gamma * y1
    dI1dt = epsilon*(1/60*(1+np.tanh((0.05-x1)/0.001)) - I1)
    dx2dt = x2**2 * (1 - x2) - y2 + I2 + theta*(vs-x2)/(1+np.exp(-lamb*(x1-q)))
    dy2dt = A * np.exp(alpha * x2) - gamma * y2
    dI2dt = epsilon*(1/60*(1+np.tanh((0.05-x2)/0.001)) - I2)

    return [dx1dt, dy1dt, dI1dt, dx2dt, dy2dt, dI2dt]
```

```python
def system(t, vars):
    x1, y1, I1, x2, y2, I2, p= vars
    dx1dt = x1**2 * (1 - x1) - y1 + I1 + theta*rho(p)*(x2 - x1)
    dy1dt = A * np.exp(alpha * x1) - gamma * y1
    dI1dt = epsilon*(1/60*(1+np.tanh((0.05-x1)/0.001)) - I1)
    dx2dt = x2**2 * (1 - x2) - y2 + I2 + theta*rho(p)*(x1 - x2)
    dy2dt = A * np.exp(alpha * x2) - gamma * y2
    dI2dt = epsilon*(1/60*(1+np.tanh((0.05-x2)/0.001)) - I2)
    dpdt = theta*(x1-x2)

    return [dx1dt, dy1dt, dI1dt, dx2dt, dy2dt, dI2dt, dpdt]
```

Time series tools

1. Hurst exponent (H): measuring persistence.
2. Sample entropy (SE): measuring complexity.
3. 0-1 test (K): measuring chaos.
4. Cross-correlation function (Γ): measuring synchrony between neurons.
5. Kuramoto order parameter (B): measuring synchrony between neurons.

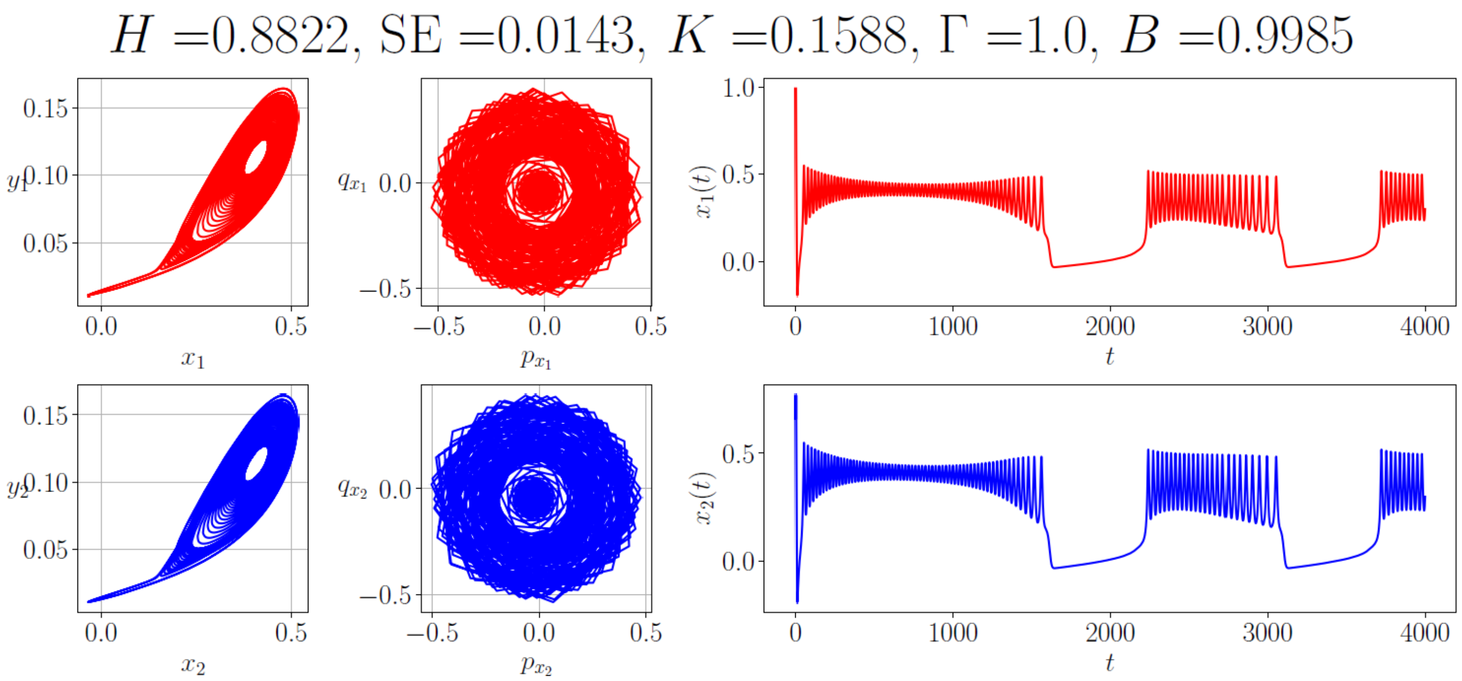


Fig. Applying various tools on the time series generated from simulating the models of coupled neurons.

Hurst exponent (Hurst, 1951)

1. Measures the long-term memory/persistence in time series.

2. Computed using rescaled-range analysis (Qian and Rasheed, 2004).

$$\mathbb{E}\left[\frac{R(t)}{S(t)}\right] = ct^H, \qquad t \to \infty$$

3. $H \in [0,1]$.

4. $H \in [0,0.5)$: anti-persistence (negative dependence on previous values), $H \approx 0.5$: random walk, $H \in (0.5,1]$: positive dependence on previous values.

Sample entropy (Richman & Moorman, 2000)

1. Assesses the complexity of time series data.

2. It is the negative natural log of the probability that if two sets of simultaneous data points of length ‘p’ have distance less than ‘ε’, then the similar thing happens to two sets of simultaneous data points of length ‘p+1’.

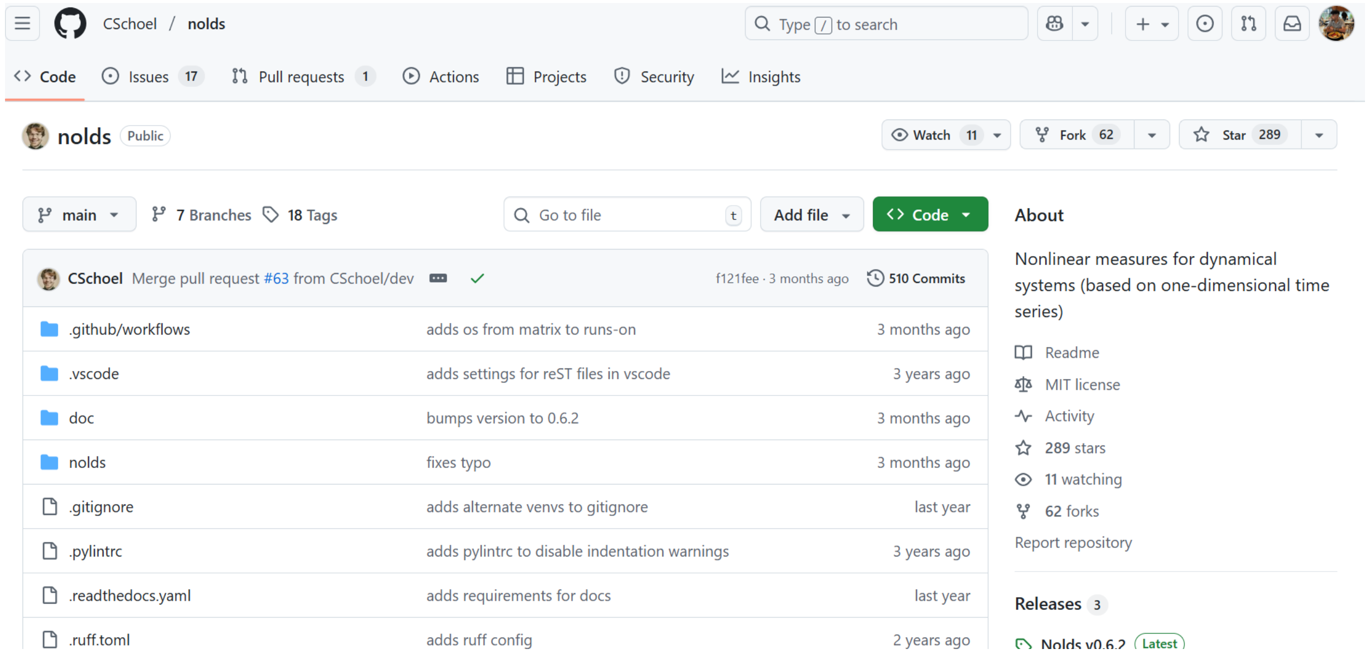
$$\mathrm{SE}(p, \varepsilon, N) = \lim_{N \to \infty} \left(-\log_e \frac{A^p(\varepsilon)}{B^p(\varepsilon)} \right)$$

3. A higher SE indicates higher complexity.

4. Can be normalised between 0 and 1.

‘nolds’ package (Scholzel, 2019)

1. Stands for ‘NOnLinear measures for Dynamical Systems’, based on .
2. Provides functions for directly implementing the rescaled-range based Hurst exponent and also sample entropy to time series.
3. Functions are `nolds.hurst_rs()` and `nolds.sampen()`.
4. Also provides other sophisticated tools for nonlinear measures.



0-1 test (Gottwald and Melbourne, 2009, 2016)

1. Compute two translated variables from the time series.

$$\tilde{p}(t;e) = \sum_{k=1}^{t} x(k)\cos(ek),$$

$$\tilde{q}(t;e) = \sum_{k=1}^{t} x(k)\sin(ek).$$

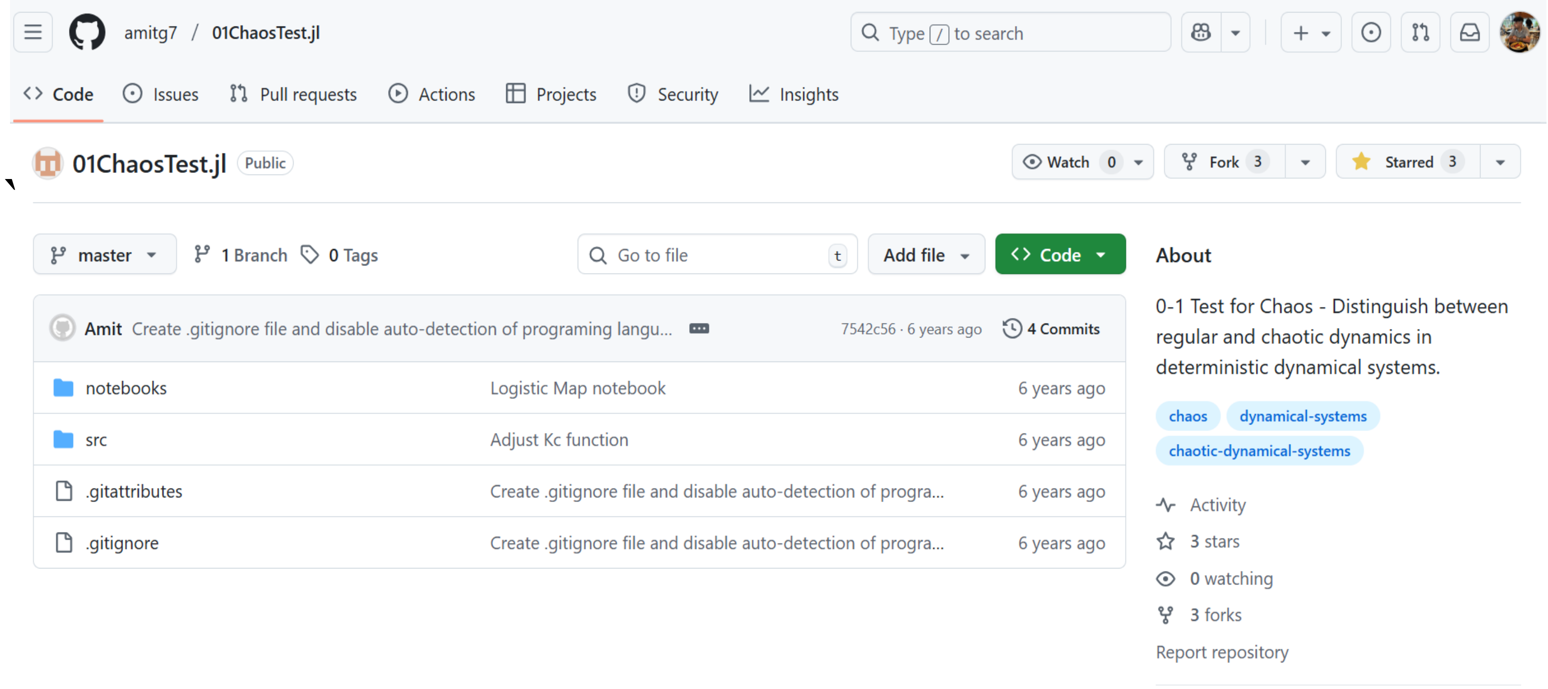
2. Then compute a mean square displacement term ‘m(t; e)’ and a correction term.

3. Compute the growth rate ‘K’ that quantifies ‘chaos’.

4. ‘K ≈ 0’ indicates regular dynamics and ‘K ≈ 1’ indicates chaos.

A julia package for 0-1 test

1. Translated to .
2. Used `curve_fit()` and `fsolve()` functions from the `scipy.optimize` suite.
3. Used `pearsonr ()` function to compute Pearson’s correlation coefficient from `scipy.stats’ suite in one of the steps.



Cross-correlation coefficient (many authors)

1. For two time series from nodes ‘i’ and ‘m’, Γ is given by

$$\Gamma_{\mathrm{i,m}} = \frac{\langle \tilde{x}_i(n)\tilde{x}_m(n)\rangle}{\sqrt{\langle(\tilde{x}_i(n))^2\rangle\langle(\tilde{x}_m(n))^2\rangle}}$$

2. The average is calculated over time and the variation from the mean is

$$\tilde{x}(n) = x(n) - \langle x(n)\rangle$$

3. $|\Gamma| = 1$ indicates total synchrony and $|\Gamma| < 1$ is asynchrony. Moreover, $\Gamma = 1$ represents in-phase synchrony and $\Gamma = -1$ represents anti-phase synchrony.

```
## cross-correlation coeff
phi_x1 = np.array(x1_sol[5000:])
phi_x2 = np.array(x2_sol[5000:])

x1_tilde = phi_x1 - np.mean(phi_x1)
x2_tilde = phi_x2 - np.mean(phi_x2)

Numerator = np.mean(x1_tilde*x2_tilde)
Denominator = np.sqrt(np.mean(x1_tilde**2)*np.mean(x2_tilde**2))

cc = Numerator/Denominator
```

Kuramoto's order parameter (Kuramoto and Battogtokh, 2002)

1. Phase of a neuron 'm' is

$$\zeta_m = \tan^{-1}\left(\frac{y_m(t)}{x_m(t)}\right)$$

2. The complex valued Kuramoto index B is then

$$B_m(t) = \exp(i\zeta_m(t)), \qquad i = \sqrt{-1}.$$

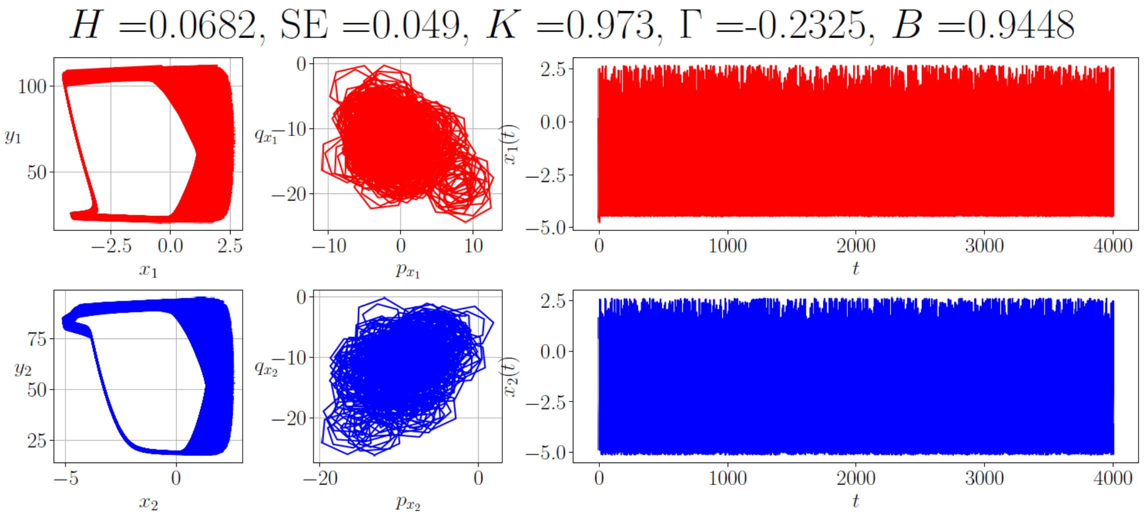
3. The index at time 't' is then

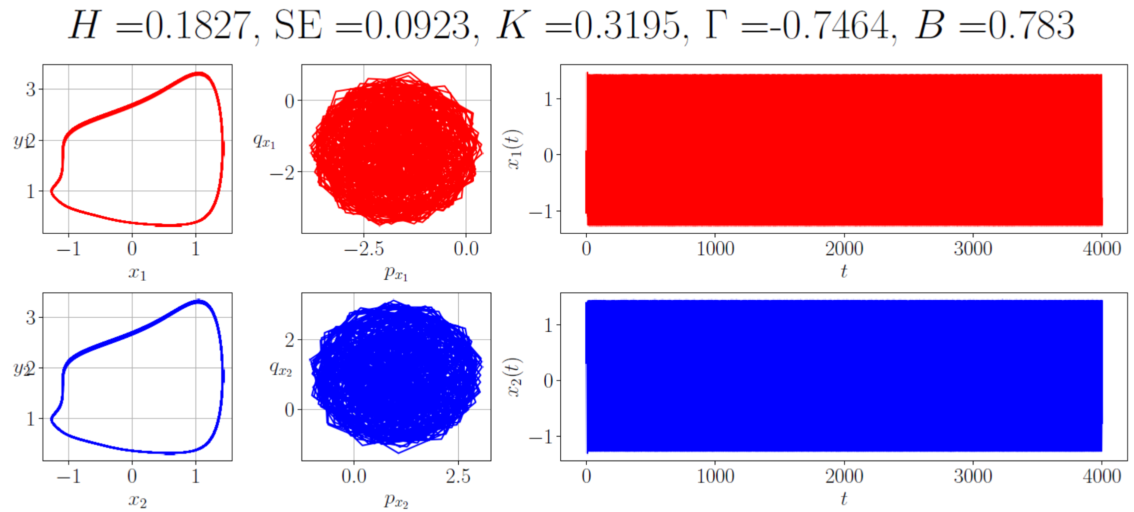
$$B(t) = \left|\frac{1}{N}\sum_{m=1}^{N} B_m(t)\right|.$$

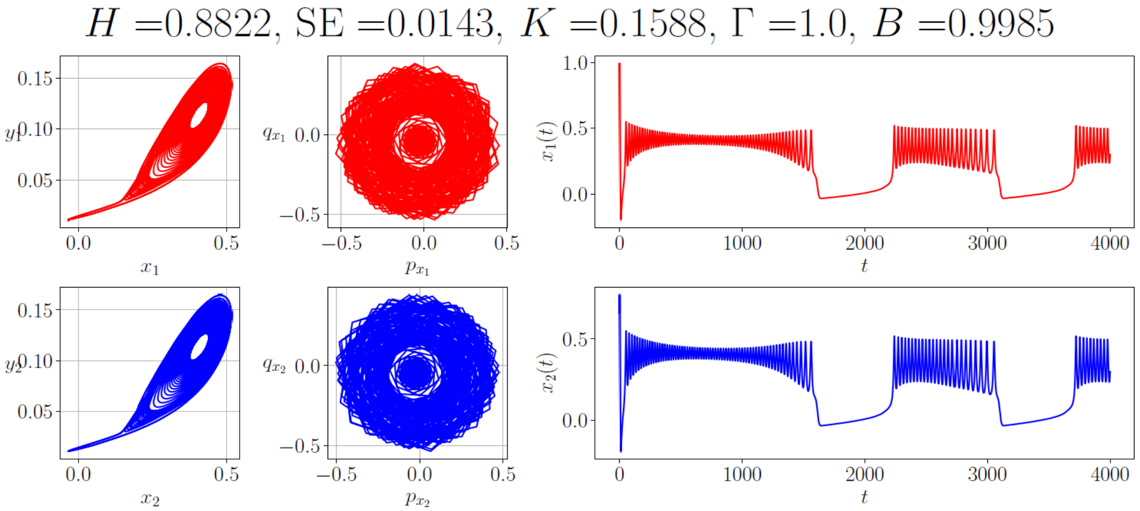
```
## Kuramoto order parameter
l1 = np.arctan(y1_sol/x1_sol)
l2 = np.arctan(y2_sol/x2_sol)

Ind1 = np.exp(1j*l1)
Ind2 = np.exp(1j*l2)
Indt = np.abs(1/2*(Ind1+Ind2))
Kuram = np.mean(Indt)
```

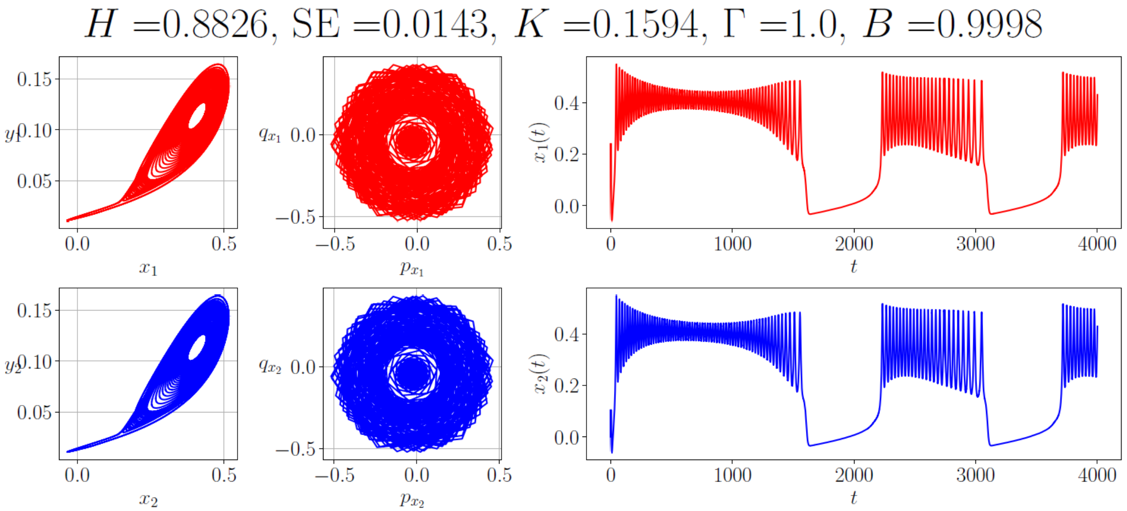
4. When B = 1, this means the nodes are all fully coherent and their phases are all locked. Any value B < 1 represents incoherence.

Results from gap-junction coupling



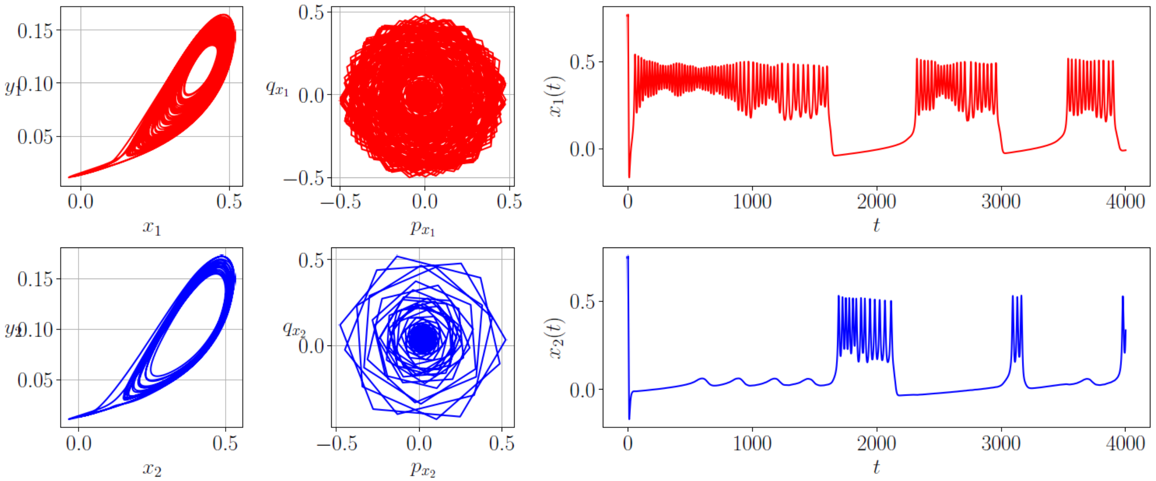




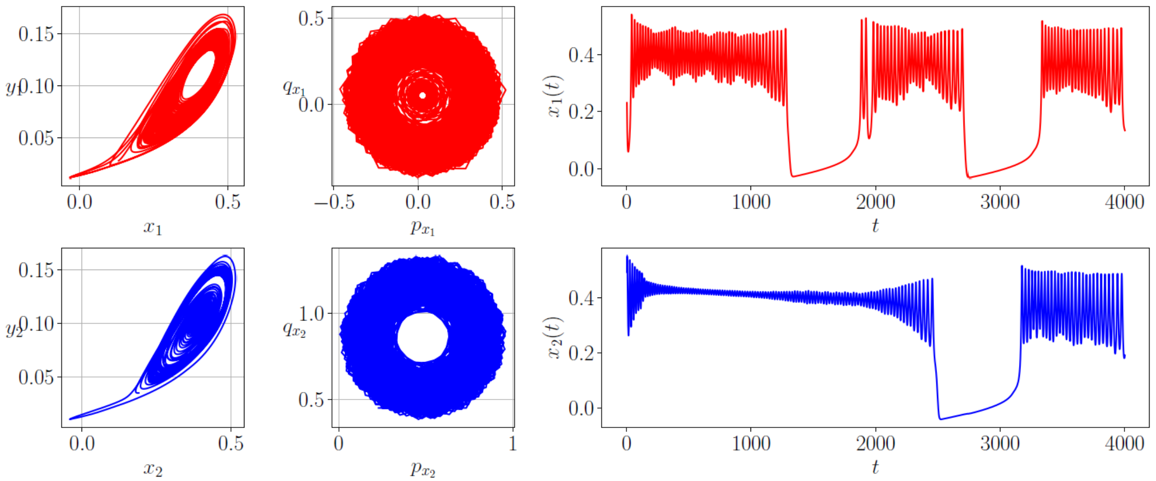


Results from chemical coupling

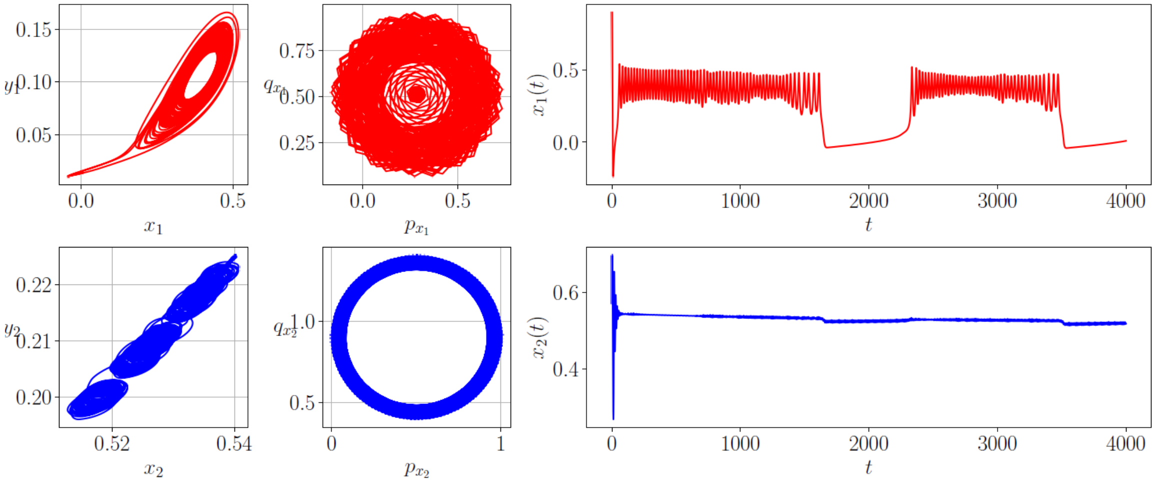
H =0.929, SE =0.0077, K =0.0934, Γ =-0.5153, B =0.9313



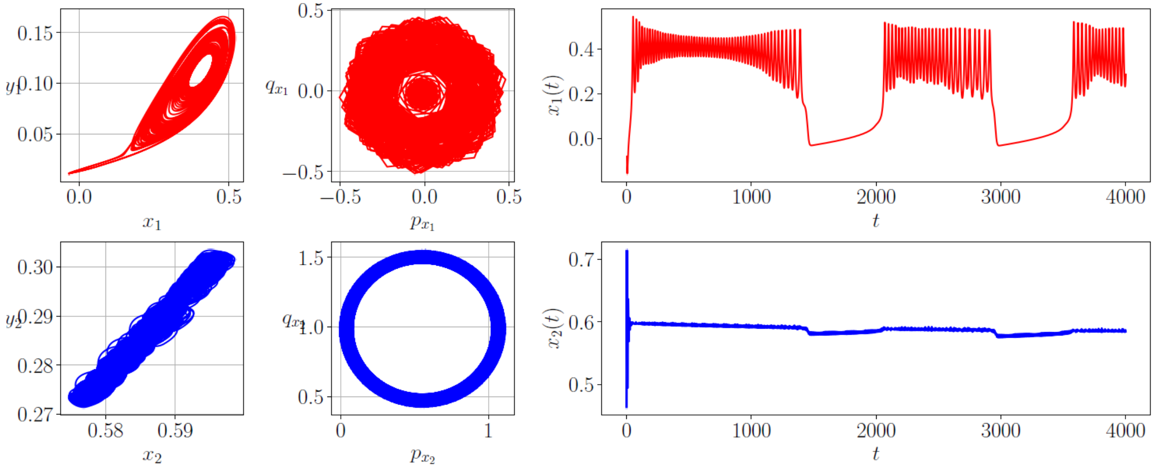
H =0.9142, SE =0.0132, K =0.1093, Γ =0.3314, B =0.9667



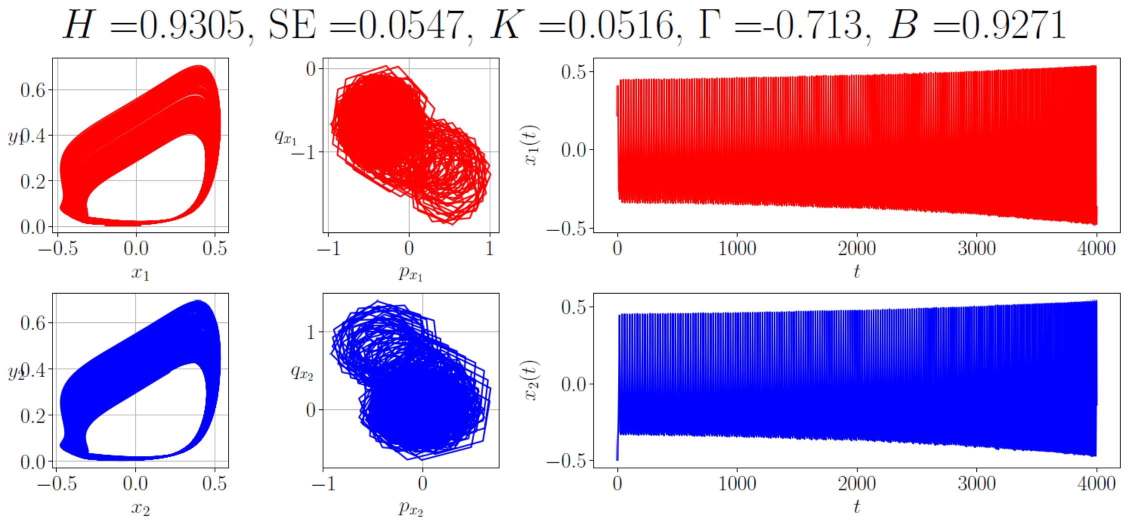
H =0.6073, SE =0.0158, K =0.0803, Γ =0.6919, B =0.9687

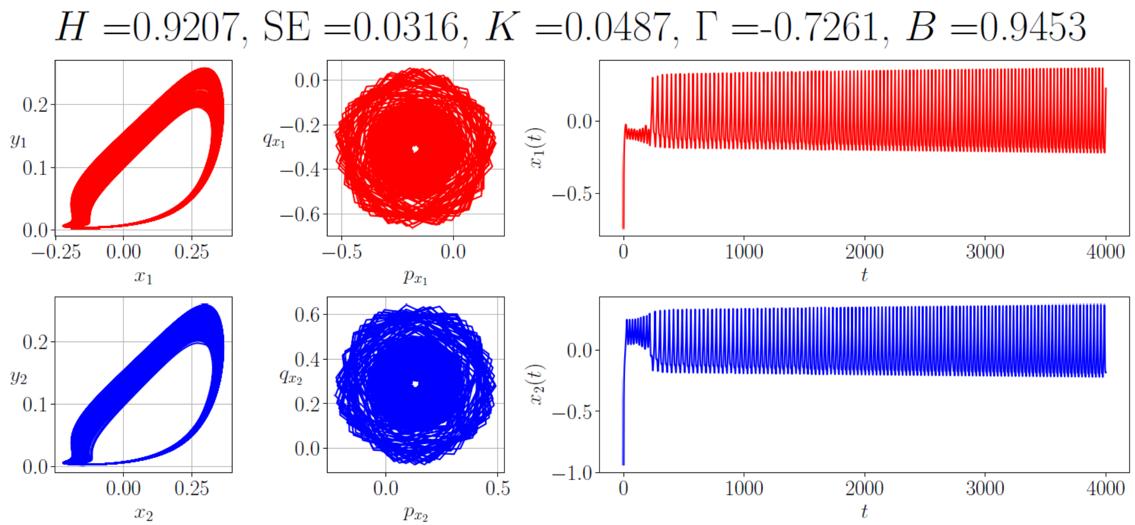


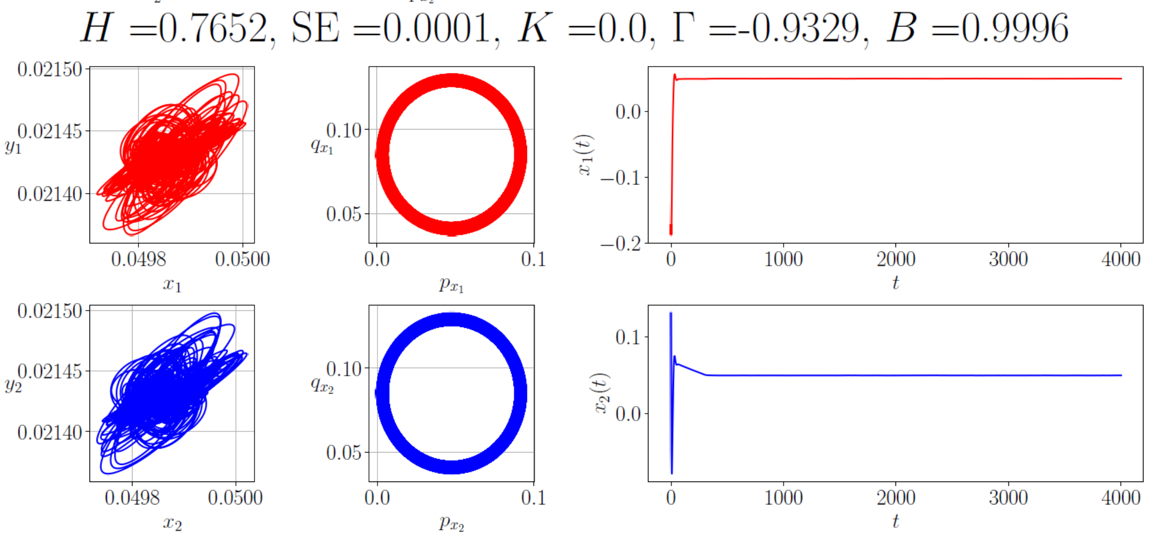
H =0.5853, SE =0.027, K =0.0851, Γ =0.7929, B =0.9654

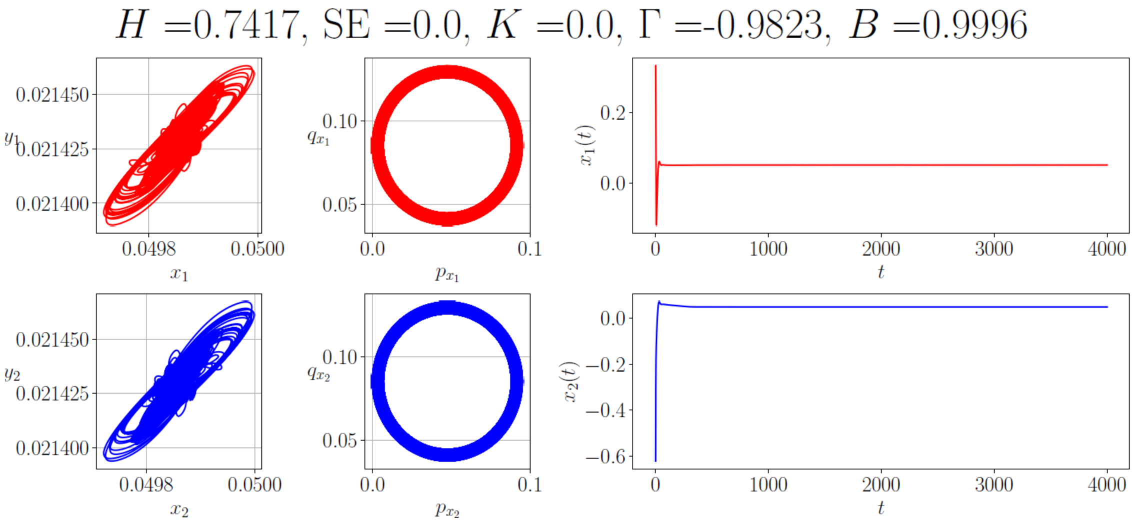


Results from electromagnetic coupling

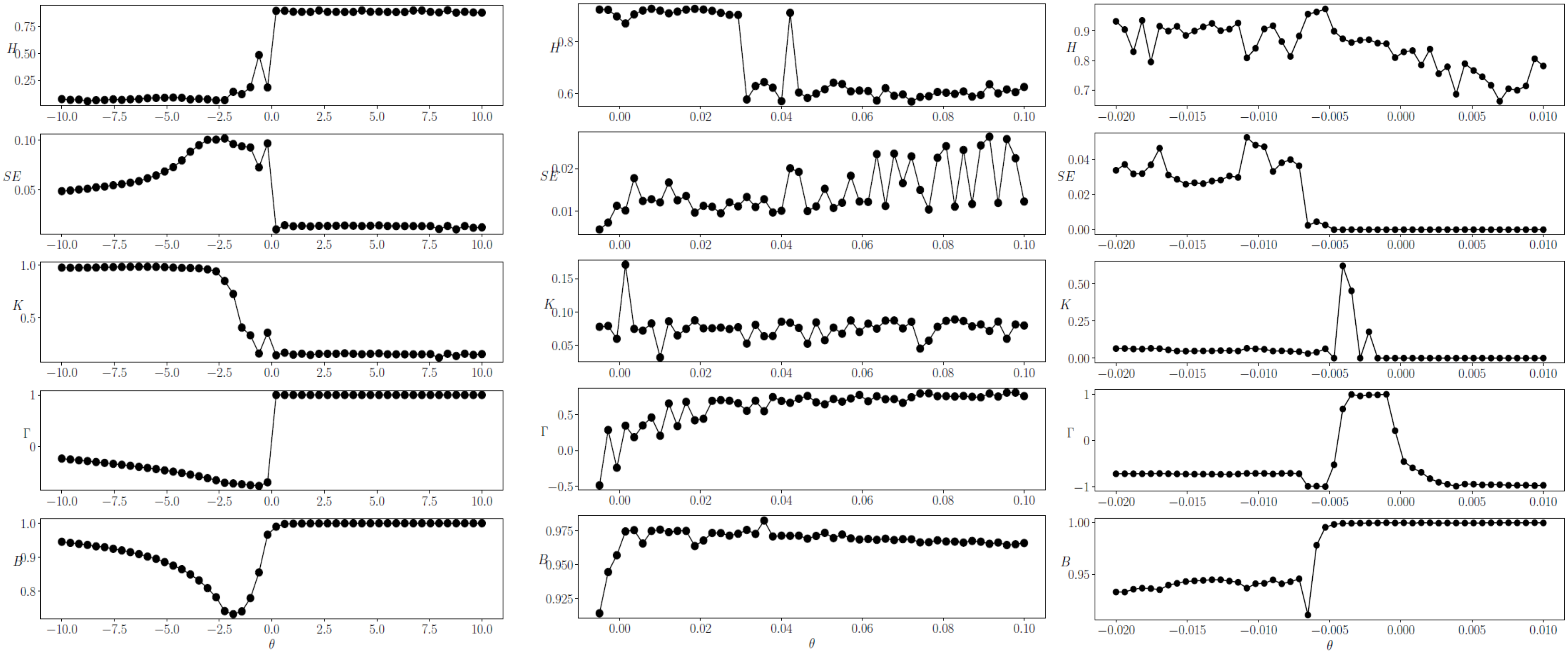








Plots with varying coupling strength



Separate experiment from visualization

1. An extra step for data processing before visualization

2. This is where comes handy! And it's worth it!

3. Read "Taming the Chaos of Computational Experiments" by T. G. Kolda for more on this: siam.org/publications/siam-news/articles/taming-the-chaos-of-computational-experiments/

```python
## Code to create the data files

SS = np.linspace(-10, 10, 50)

count = 1
HH=[]
SE=[]
KKTest = []
CC = []
Kuramoto = []
for theta in SS:
    print("count = "+str(count))
    x1_sol, y1_sol, x2_sol, y2_sol, tt, cc, KK1, KK2, Kuram, h1, h2, se1, se2 = bif_gap(theta)
    HH+=[(h1+h2)/2, ]
    SE+=[(se1+se2)/2, ]
    CC+=[cc, ]
    KKTest+=[(KK1+KK2)/2, ]
    Kuramoto+=[Kuram, ]

    print("H=", (h1+h2)/2)
    print("SE=", (se1+se2)/2)
    print(" ")

    count+=1

df = pd.DataFrame({
    'theta': SS,
    'H': HH,
    'SE': SE,
    'CC': CC,
    'KK': KKTest,
    'Kuramoto': Kuramoto
})

## Create the data file
df.to_csv('data_gap.csv', index=False)
```

Visualization using

In [6]:

```
## Load the data file
dfGap = pd.read_csv('data_gap.csv')
dfGap
```

Out[6]:

	theta	H	SE	CC	KK	Kuramoto
0	-10.000000	0.075755	0.048965	-0.230950	0.974990	0.945371
1	-9.591837	0.068889	0.049600	-0.247435	0.973426	0.942219
2	-9.183673	0.070847	0.050584	-0.264226	0.974289	0.939167
3	-8.775510	0.056358	0.051361	-0.279553	0.974688	0.936165
4	-8.367347	0.065453	0.052763	-0.297540	0.976047	0.931959
5	-7.959184	0.067204	0.053522	-0.314207	0.979496	0.929517
6	-7.551020	0.073805	0.054817	-0.332759	0.980077	0.924822
7	-7.142857	0.068470	0.056010	-0.350708	0.981813	0.920045
8	-6.734694	0.074268	0.057403	-0.369906	0.982537	0.914924
9	-6.326531	0.075233	0.058941	-0.391002	0.983551	0.909313
10	-5.918367	0.083993	0.061830	-0.412140	0.982661	0.901997

```
sz=18
%matplotlib notebook
matplotlib.rc('xtick', labelsize=sz)
matplotlib.rc('ytick', labelsize=sz)

SS = np.linspace(-10, 10, 50)

fig, axs = plt.subplots(5,1, figsize=(10, 12))

axs[0].set_ylabel('$H$',rotation=False, fontsize=sz)
axs[1].set_ylabel('$SE$',rotation=False, fontsize=sz)
axs[2].set_ylabel('$K$',rotation=False, fontsize=sz)
axs[3].set_ylabel('$\\Gamma$',rotation=False, fontsize=sz)
axs[4].set_ylabel('$B$',rotation=False, fontsize=sz)
axs[4].set_xlabel('$\\theta$', fontsize=sz)

HH = dfGap['H']
SE = dfGap['SE']
KKTest = dfGap['KK']
CC = dfGap['CC']
Kuramoto = dfGap['Kuramoto']


axs[0].plot(SS, HH, 'ko-', ms=10)
axs[1].plot(SS, SE, 'ko-', ms=10)
axs[2].plot(SS, KKTest, 'ko-', ms=10)
axs[3].plot(SS, CC, 'ko-', ms=10)
axs[4].plot(SS, Kuramoto, 'ko-', ms=10)

plt.tight_layout()
```

Summary

1. Coupling induces ‘chaos’ in the inhibitory regime.

2. Excitatory coupling and its more positive values drive the coupled system into exhibiting bursting. Also, both neurons synchronize.

3. In electromagnetic coupling, excitatory coupling drives the system to decay oscillation, falling into a symmetric equilibrium point.

4. Future work: move to GPU accelerated framework using `CuPy`:

5. Future work: use a complex network of neuron and integrate `NetworkX`:

Acknowledgements

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UCD DUBLIN
University College Dublin

DIGITAL UNIVERSITY KERALA
Curating a responsible digital world

Reference: Indranil Ghosh, Hammed Olawale Fatoyinbo, and Sishu Shankar Muni, "Time series analysis of coupled slow-fast neuron models: From Hurst exponent to Granger causality" (2025), arxiv.org/abs/2507.13570.

repository: github.com/indrag49/TS-SlowFast-dML